

A STUDY OF SAMPLING THEOREM CONSTRAINTS

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the Faculty of the Graduate Division

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SUMMARY

It has long been known that a band limited function can be recovered if only certain samples of the function are available. The ability of the function to be represented by, and recovered from, a train of samples apparently has not been discussed thoroughly in any single article in the literature. The purpose of this study is to express as clearly as possible the constraints which must be met in order that such a function can be reconstructed from a train of its samples.

Although the usual statement of the sampling theorem covers only periodic, equispaced samples of the function itself, methods involving samples taken in clusters, samples of the derivatives of the desired function, and samples of bandpass functions have recently been presented in the literature. This research includes a study of sampling and recovery for these methods.

The spectrum of a non-periodic $f(t)$ is defined as its Fourier transform, and the spectrum of a periodic $s(t)$ is defined by the coefficients of its Fourier series expansion. Only those non-periodic $f(t)$ which have Fourier transforms are considered. Using these concepts of frequency spectra, conditions sufficient for recovery of a function sampled with pulses of non-zero time duration are stated.

The operation of impulse sampling is defined, and sufficient conditions for determination and recovery are developed. This is done by expanding the exponential $e^{j\omega t}$ in the inverse Fourier transform of $f(t)$ (band limited to B cps) into a Fourier series in ω valid in the

region $-2\pi B \leq \omega \leq 2\pi B$, and interpreting the result as the response of an ideal low pass filter to a train of impulses.

The existence of a certain minimum rate of taking periodic equally-spaced samples necessary for determination of a function is proved for instantaneous samples. This is shown to be the Nyquist rate.

Information available in the literature on derivative and cluster sampling is expanded. A unified approach to the constraints for determination and recovery for both methods is presented. The spectrum of each sample train is shown to have several components at any frequency ω , assuming sampling below the Nyquist rate. The spectra of the samples of the derivative, or of the other samples in a cluster, are shown to differ from the spectrum of the samples of the original function so that any desired component can be separated by a network of proper design. The essence of the procedure is solution of simultaneous equations which have the separating networks $I_k(\omega)$ as unknowns. Two examples are included to clarify the methods.

It is concluded that further work could be done to show that the filter required for recovery from impulse samples is not uniquely related to the sampling rate. Also, further investigation might be made into the implication that sampling a function with non-zero time duration pulses at any rate arbitrarily below the Nyquist rate can, under certain circumstances, determine the function uniquely.

CHAPTER I

INTRODUCTION

Given a function f defined on a domain D , sampling of the function can be considered as the process of defining a new function on a subset D_s of the domain D . The sample function f_s has values identical to those of the original function on this subset.

The area of mathematics which concerns the relationship between f and f_s is sampling theory. The mathematical operation of determining the values of the function f from the sample function f_s is called interpolation.

Restrictions on the method of determining the sample values used and on the reconstruction of the original function from them are, in sampling theory, the basic problems that concern the communication art.

The ability of certain continuous functions to be represented by a sequence of samples allows the use of the various types of pulse modulation in the transmission of these functions. These systems offer the opportunity for time multiplexing of signals, as well as noise advantages and more efficient use of spectrum space in some cases.¹

The first known statement about sampling theory was made by Cauchy in 1841:

If a signal is a magnitude-time function, and if time is divided into equal parts forming subintervals such that each subdivision comprises an interval T seconds long, where T is less than half the period of the highest significant frequency component of the signal; and if one instantaneous sample is taken from each

subinterval in any manner; then a knowledge of each sample plus a knowledge of the instant within each subinterval at which each sample is taken contains all of the information of the original signal.²

A precise statement of the basic principle was first introduced into electrical engineering literature by Shannon in 1949: "If a function $f(t)$ contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart."³

Shannon used this theorem in developing his formula for maximum error-free channel capacity. It will be noted that he specifies the position of the points exactly, rather than saying that they fall arbitrarily in some interval, as Cauchy did.

In the following decade this basic statement was extended to include derivative,⁴ cluster,⁵ and narrow band sampling.⁶ However, a single precise expression of the sampling theorem including these special cases has not yet been presented.

This study considers the special cases mentioned above in addition to the basic statement of the requirements for sampling and recovery of a band limited function. It is assumed throughout that the function sampled is real and that it can be represented by its Fourier integral transform. Thus, random processes are not included. On the basis of these assumptions and others, which will be stated when they are made, constraints sufficient for determination and recovery of a function from a set of samples are developed.

CHAPTER II

SAMPLING AND RECOVERY OF A BAND-LIMITED FUNCTION WITH PULSES OF FINITE AMPLITUDE AND NON-ZERO TIME DURATION

2.1 Procedure.--The constraints for the operation described in the title will be developed by considering the multiplication of a band-limited non-periodic function $f(t)$ by a periodic pulse train.

2.2 Spectrum of a Periodic Function.--If a function of time $s(t)$ is periodic, with period T , $s(t) = s(t \pm nT)$, $n = 0, 1, 2, \dots$. Its frequency spectrum can be interpreted as a set of impulses in the ω (radians per second) or f (cycles per second) domain occurring at $1/T$ cps (cycles per second) intervals.

If the m^{th} impulse is the one occurring at m/T cps ($m = 0, 1, 2, \dots$) its value, c_m , which may be complex, is found by evaluating the integral

$$c_m = \int_{t_0}^{t_0+T} s(t) e^{-j\omega_m t} dt, \quad \omega_m = \frac{2m\pi}{T}. \quad (1)$$

The t_0 is an arbitrary point in the time domain.

The c_m found in (1) is the coefficient in the exponential Fourier series expansion of $s(t)$ which is

$$s(t) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} c_m e^{j\omega_m t}. \quad (2)$$

Thus, the frequency spectrum of $s(t)$, a periodic function, is defined as a set of impulses in the ω domain occurring at $2\pi/T$ rps (radians per second) intervals.⁷ The value of the m^{th} impulse is the m^{th} coefficient in the Fourier series expansion of $s(t)$.

2.3 Spectrum of a Non-Periodic Function--The frequency spectrum of a non-periodic function $f(t)$ can be defined as the Fourier transform $F(\omega)$.⁸

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt, \quad -\infty < \omega < +\infty. \quad (3)$$

This transform exists provided the condition

$$\int_{-\infty}^{+\infty} |f(t)| dt < \infty \quad (4)$$

is met. This is a sufficient, but not necessary, condition for the existence of the Fourier transform of $f(t)$ as defined by Equation (3). Some functions may exist for which an $F(\omega)$ can be found, but for which Equation (4) does not converge.

A function $F(\omega)$ is band-limited if a positive number B exists such that $F(\omega)$ is zero for all $|\omega|$ greater than $2\pi B$ rps. In addition, it can be defined as a low pass function if there is no positive number $X < B$ such that $F(\omega)$ is zero for all $|\omega|$ less than $2\pi X$. The smallest value of B for which the statement above is true will be designated B_n .

The frequency spectrum of a typical band-limited low pass function may look like Figure 1.

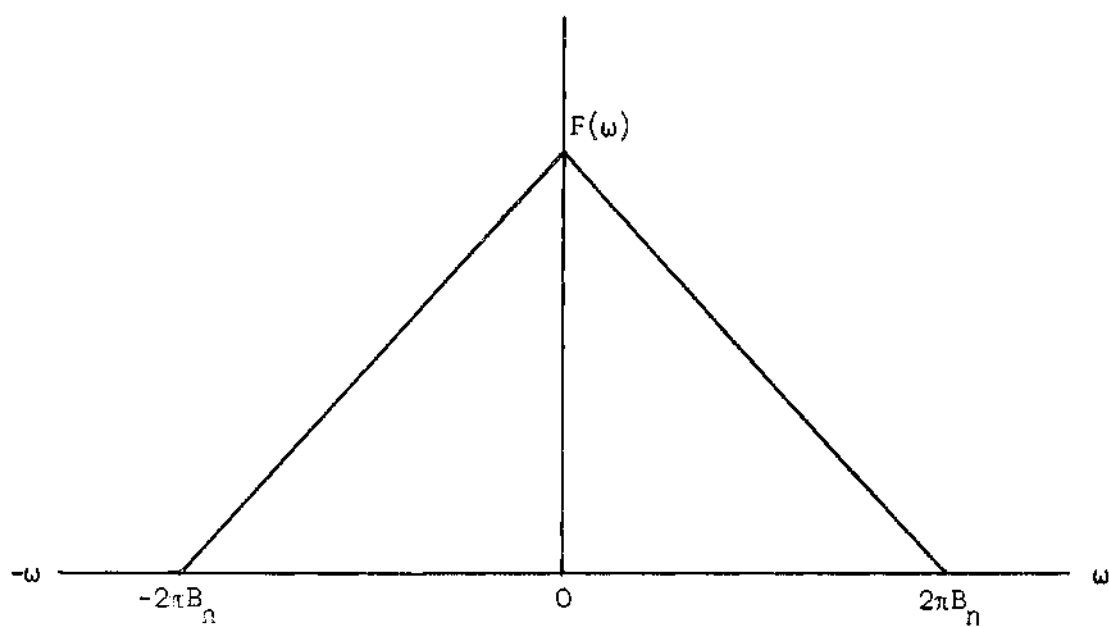


Figure 1. Typical Band-Limited
Low Pass Function

2.4 Sampling with Periodic Pulses.--Examine a train of periodic pulses having fundamental angular frequency $\omega_s = 2\pi/T$, arbitrary but non-zero duration τ , and amplitude A . This train is shown in Figure 2.

This function will be used as the sampling function for the type of sampling described in this chapter. The Fourier series of this pulse train is given by the expression

$$s(t) = \frac{A\tau}{T} \sum_{n=-\infty}^{+\infty} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} e^{j\omega_n t}, \quad \omega_n = n\omega_s. \quad (5)$$

The frequency spectrum is shown in Figure 3.

If one multiplies the original band-limited $f(t)$ by this sampling function $s(t)$, the product is zero when $s(t)$ is zero and is $Af(t)$ when $s(t)$ is A . This product generates a train of non-zero time duration samples of $f(t)$.¹⁰

$$s(t) \times f(t) = f(t) \frac{A\tau}{T} \sum_{n=-\infty}^{+\infty} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} e^{j\omega_n t}. \quad (6)$$

Since $f(t)$ is independent of the summation, it can be taken inside the summation sign.

$$s(t) \times f(t) = \frac{A\tau}{T} \sum_{n=-\infty}^{+\infty} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} f(t) e^{j\omega_n t}. \quad (7)$$

The Fourier transform of $f(t) \times s(t)$ can be called $F_s(\omega)$.

$$F_s(\omega) = \int_{-\infty}^{+\infty} f(t) \times s(t) e^{-j\omega t} d\omega. \quad (8)$$

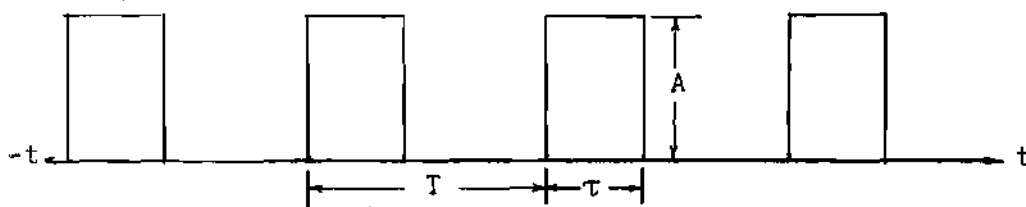


Figure 2. Train of Periodic Pulses

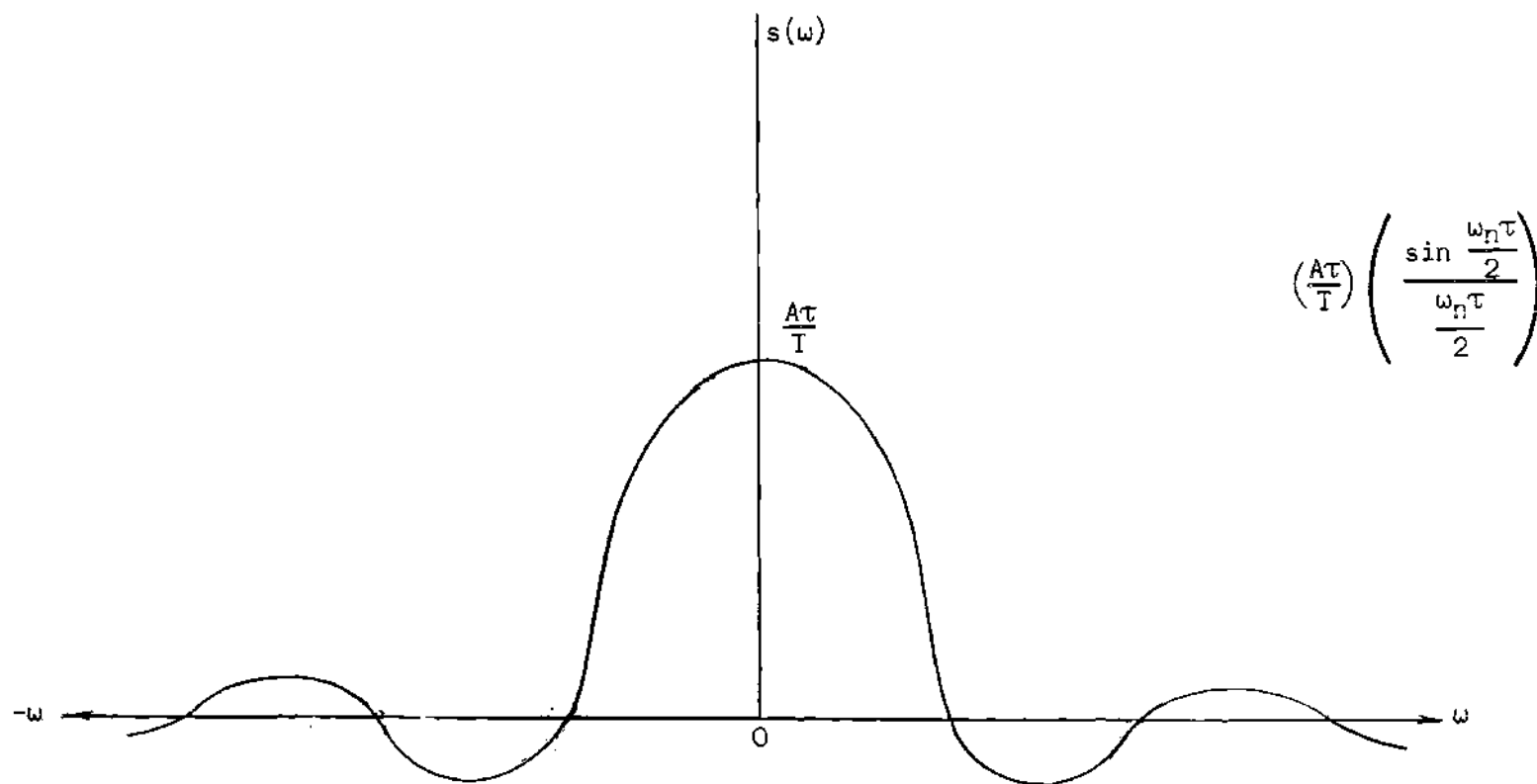


Figure 3. Frequency Spectrum of Sampling Function $s(t)$

$$F_s(\omega) = \int_{-\infty}^{+\infty} \frac{A\tau}{T} \sum_{n=-\infty}^{n=+\infty} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} f(t) e^{j\omega_n t} e^{-j\omega t} d\omega. \quad (9)$$

If it is assumed that the order of integration and summation can be changed, the series can be integrated term by term.²

$$F_s(\omega) = \frac{A\tau}{T} \sum_{n=-\infty}^{n=+\infty} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} \int_{-\infty}^{+\infty} f(t) e^{+j\omega_n t} e^{-j\omega t} dt, \quad (10)$$

$$F_s(\omega) = \frac{A\tau}{T} \sum_{n=-\infty}^{n=+\infty} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} \int_{-\infty}^{+\infty} f(t) e^{-jt(\omega - \omega_n)} dt, \quad (11)$$

$$F_s(\omega) = \frac{A\tau}{T} \sum_{n=-\infty}^{n=+\infty} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} F(\omega - \omega_n). \quad (12)$$

According to this last equation (12) there is a replica of $F(\omega)$ centered at the location of each line in the spectrum of the sampling function $s(t)$, and multiplied by the factor A .

$$\frac{A\tau}{T} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} \quad (13)$$

The spectra of the sampling function of Figure 2 and the sampled function of Figure 1 are shown in Figures 3 and 4, assuming that

$$\omega_s > 4\pi B_n.$$

1.5 Recovery of the Sampled Function.-- $F(\omega)$ can be recovered from $F_s(\omega)$ if the sequence of samples is applied to a filter passing all frequencies $|\omega| \leq 2\pi B_n$ equally, and completely rejecting all frequencies $|\omega| \geq \omega_s - 2\pi B_n$. Also, it can be seen from Figure 4 that the requirements on the filter become less and less stringent as the sampling frequency increases.

At the minimum ω_s of $4\pi B_n$ an ideal low pass filter is required. As ω_s increases, the filter requirements become less stringent. A "guard band" develops in which the behavior of the filter is arbitrary.¹¹ If ω_s is much larger than $4\pi B_n$, any kind of "cheap and dirty" low pass filter will often do. In the limit as ω_s becomes very high, continuous "samples" are taken; the complete function is being sampled and no filter is necessary.

When ω_s is less than $4\pi B_n$, the spectra of Figure 4 overlap and it is not possible to recover $F(\omega)$ with a single low pass filter. It can be seen from Figure 4 that $4\pi B_n$ is the lowest value of ω_s for which recovery can be effected in this way. This particular ω_s , which is twice the highest frequency of interest in a band-limited signal, is called the "Nyquist frequency," or "Nyquist rate," after H. Nyquist of Bell Telephone Laboratories.¹²

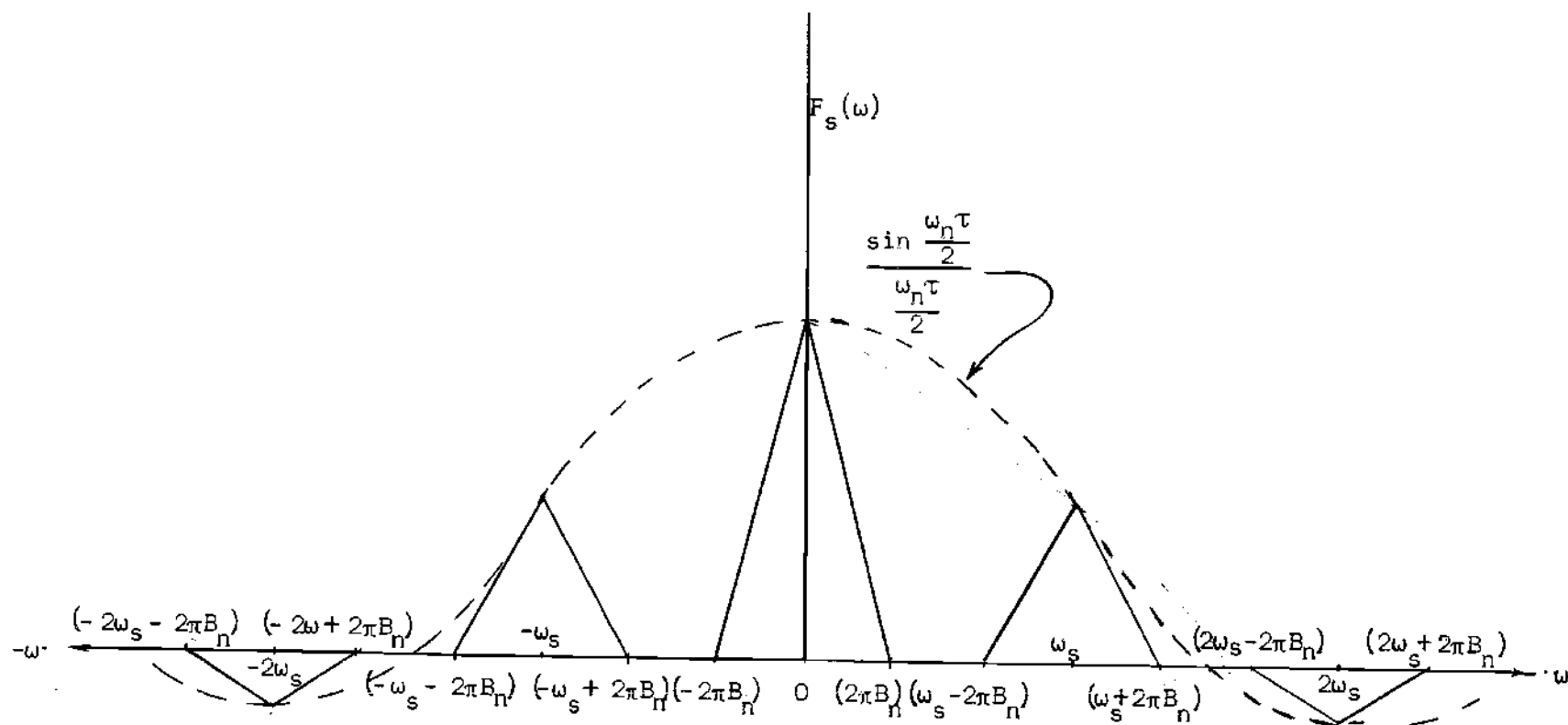


Figure 4. Frequency Spectrum of Sampled Function

CHAPTER III

SAMPLING AND RECOVERY OF A BAND LIMITED FUNCTION WITH IMPULSES

3.1 Sampling the Function--The basis of the calculations in this chapter were developed by Balakrishnan.¹³ It is assumed that the band-limited function $f(t)$ can be represented by the following integral:

$$f(t) = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} F(\omega) e^{j\omega t} d\omega, \quad B > 0. \quad (14)$$

In the interval $-2\pi B \leq \omega \leq 2\pi B$, $e^{j\omega t}$ is a finite continuous function with a finite number of maxima and minima. It is therefore expandable in a Fourier series. A function $G(\omega)$ can be defined in the following manner:

$$G(\omega) = e^{j\omega t}, \quad |\omega| \leq 2\pi B, \quad (15a)$$

$$G(\omega) = 0, \quad |\omega| > 2\pi B. \quad (15b)$$

$G(\omega)$ can be extended as a periodic function $G_e(\omega)$ having period $4\pi B$. This extended function can be expressed as a Fourier series in ω .

$$G_e(\omega) = \frac{1}{T} \sum_{m=-\infty}^{m=+\infty} g_m e^{j\omega_m \omega}, \quad \begin{aligned} T &= 4\pi B \\ \omega_m &= \frac{2\pi m}{4\pi B} = \frac{m}{2B} \end{aligned} \quad (16)$$

$$g_m = \int_{-2\pi B}^{+2\pi B} G_e(\omega) e^{-j\frac{m\omega}{2B}} d\omega. \quad (17)$$

But $G_e(\omega) = e^{j\omega t}$ in the interval $-2\pi B, 2\pi B$. Therefore:

$$g_m = \int_{-2\pi B}^{2\pi B} e^{j\omega t} e^{-j\frac{m\omega}{2B}} d\omega = \int_{-2\pi B}^{2\pi B} e^{j\omega(t - \frac{m}{2B})} d\omega. \quad (18)$$

If the integral is evaluated

$$g_m = \frac{2}{t - \frac{m}{2B}} \sin 2\pi B(t - \frac{m}{2B}).$$

If g_m is substituted into Equation (16)

$$G_e(\omega) = \frac{1}{4\pi B} \sum_{m=-\infty}^{+\infty} \frac{2}{t - \frac{m}{2B}} \sin 2\pi B(t - \frac{m}{2B}) e^{j\frac{m\omega}{2B}}, \quad (19)$$

$$G_e(\omega) = e^{j\omega t} = \sum_{m=-\infty}^{m=+\infty} \frac{\sin 2\pi B(t - \frac{m}{2B})}{2\pi B(t - \frac{m}{2B})} e^{j\frac{m\omega}{2B}} d\omega, \quad |\omega| \leq 2\pi B \quad (20)$$

Then, if this expression for $e^{j\omega t}$ is substituted into Equation (14),

$$f(t) = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} F(\omega) \sum_{m=-\infty}^{m=+\infty} \frac{\sin 2\pi B(t - \frac{m}{2B})}{2\pi B(t - \frac{m}{2B})} e^{j\frac{m\omega}{2B}} d\omega. \quad (21)$$

Since $e^{j\omega t}$ is absolutely continuous in the interval $-2\pi B, 2\pi B$, its Fourier series representation converges boundedly to $e^{j\omega t}$ in this

interval. The order of integration and summation may therefore be changed.¹³

$$f(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{m=+\infty} \frac{\sin 2\pi B(t - \frac{m}{2B})}{2\pi B(t - \frac{m}{2B})} \int_{-2\pi B}^{+2\pi B} F(\omega) e^{j\omega \frac{m}{2B}} d\omega. \quad (22)$$

But, by Equation (14),

$$f(\frac{m}{2B}) = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} f(\omega) e^{j\omega \frac{m}{2B}} d\omega. \quad (23)$$

Therefore $f(t)$ is equal to the summation

$$f(t) = \sum_{m=-\infty}^{m=+\infty} \frac{\sin 2\pi B(t - \frac{m}{2B})}{2\pi B(t - \frac{m}{2B})} f(\frac{m}{2B}). \quad (24)$$

2.2 Recovery of the Sampled Function.--The expression

$$2B \frac{\sin 2\pi B(t - \frac{m}{2B})}{2\pi B(t - \frac{m}{2B})} \quad (25)$$

can be shown to be the time response of an ideal low pass filter having bandwidth $2\pi B$ rps to a unit impulse applied at $t = m/2B$ seconds.¹⁴

The transfer function of such a filter is shown in Figure 5. If $K(\omega)$ is the transfer function of the filter, its time response $k(t)$ to a unit impulse occurring at time t_0 is the inverse Fourier transform of $K(\omega) e^{-j\omega t_0}$.

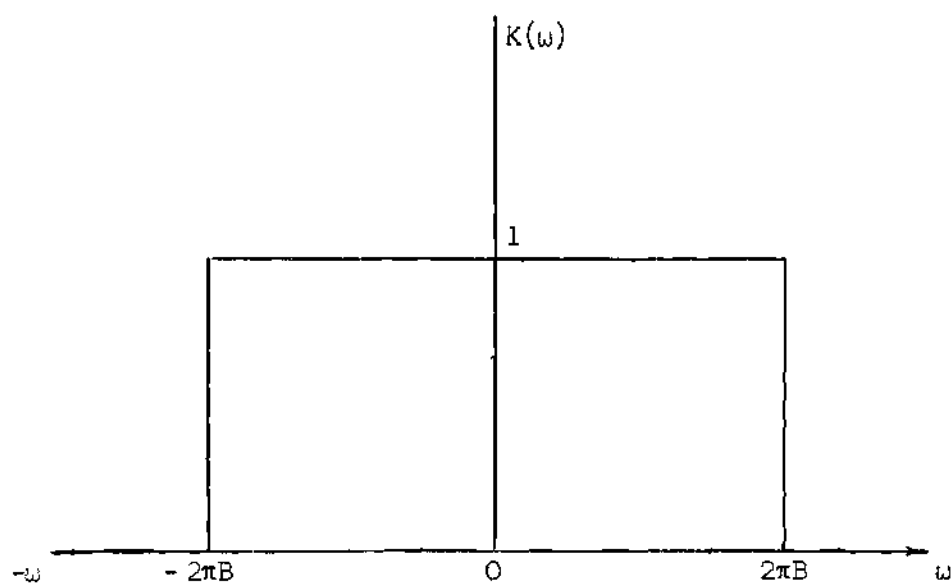


Figure 5. Transfer Function of Ideal Low Pass Filter

$$k(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K(\omega) e^{-j\omega t_0} e^{j\omega t} d\omega \quad (26)$$

$$= \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} e^{-j\omega t_0} e^{j\omega t} d\omega \quad (27)$$

$$= \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} e^{j\omega(t-t_0)} d\omega \quad (28)$$

$$= \frac{1}{2\pi j(t-t_0)} [e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}] \quad (29)$$

$$K(t) = 2B \frac{\sin 2\pi B(t-t_0)}{2\pi B(t-t_0)} \quad (30)$$

The $f(t)$ in Equation (24) can thus be interpreted as the response of an ideal low pass filter, having bandwidth $2\pi B$ rps, to a sequence of impulses spaced $1/2B$ seconds apart, the m^{th} impulse having a value $\frac{f(m/2B)}{2B}$.

At the m^{th} instant of sampling, $t_m = m/2B$. In Equation (30), $t = t_0 = t_m$. At this instant $k(t_0) = 2B$. $f(t)$ in Equation (24) is equal to $f(m/2B)$ at that particular instant. The other sampling impulses, the ones occurring at $t = (m \pm 1)/2B$, $(m \pm 2)/2B$, etc., have no effect on the output since $k(t_0)$ is zero for those values of t .

CHAPTER IV

COMPARISON OF SAMPLING WITH PULSES AND IMPULSES

4.1 Non-Zero Time Duration Pulses.--According to the development of Chapter II, sufficient constraints for sampling with non-zero time duration pulses were:

A. That the sampling and the sampled functions have a frequency spectrum definable by the Fourier series (for the periodic sampling function) and the Fourier integral (for the non-periodic, sampled function.)

B. That the sampled function $f(t)$ be such that the operations of integration and summation be interchangeable in equation (9).

Under these constraints it was shown that a sampled function is recoverable with a single low pass filter provided the sampling frequency is higher than twice the maximum frequency of the function (designated $2\pi B_n$).

If ω_s is greater than $4\pi B_n$, the transfer function of the filter is arbitrary in the interval $2\pi B < \omega < -2\pi B$, zero when ω is greater than $\omega_s - 2\pi B$, and a non-zero constant when ω is equal to or less than $2\pi B$. If ω_s is exactly $4\pi B_n$, the Nyquist rate, an ideal lowpass filter of the type described in Chapter II is required.

4.2 Impulses.--According to the development in Chapter III, a sufficient constraint for sampling $f(t)$ with impulses was that the following equality be true for all t .

$$f(t) = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} f(\omega) e^{j\omega t} d\omega. \quad (14)$$

Under this assumption, it was shown that $f(t)$ could be recovered if impulses occurring at a sufficiently large rate $\omega_s = 4\pi B$ (each impulse weighted according to the magnitude of $f(t)/2B$ at the time of its occurrence) were applied to an ideal low pass filter having a bandwidth $\omega_s/2$ rps.

The minimum ω_s is the minimum value of $4\pi B$ allowed in Equation (14) designated $4\pi B_n$. If ω_s is the minimum, then the sampling is done at the Nyquist rate. If the maximum frequency of $F(\omega)$ is less than $2\pi B$ rps, the sampling is above the Nyquist rate. It will be shown that $f(t)$ is not generally recoverable if ω_s is less than $4\pi B_n$.

4.3 Relationship Between Pulse and Impulse Samples.--In Chapter II it was shown that

$$F_s(\omega) = \frac{A\tau}{T} \sum_{n=-\infty}^{n=+\infty} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} F(\omega - \omega_n) \quad (12)$$

for sampling with pulses. If τ becomes very small, and A very large, under the restriction that $A\tau = T$.

$$F_s(\omega) \approx \sum_{n=-\infty}^{n=+\infty} F(\omega - \omega_n), \quad \text{since} \quad \lim_{\tau \rightarrow 0} \frac{\sin \frac{\omega_n \tau}{2}}{\frac{\omega_n \tau}{2}} = 1. \quad (31)$$

This expression indicates that all replicas of $F(\omega)$ have the same magnitude.¹⁵ Since τ is small, and if $f(t)$ is continuous, $f(t)$

approaches $f(m/2B)$ during the sampling interval $t = m/2B + \Delta t$. ($0 < \Delta t < \tau$) In the limit, the samples approach a train of weighted impulses of value $[A\tau f(m/2B)]$. (Impulses can be considered a limiting form of a pulse as duration (τ) becomes very small and amplitude (A) becomes very large. In this process $A\tau$, the area of the pulse, remains constant. This constant is defined to be the "value" of the impulse.) Inspection of the spectrum $F_s(\omega)$ shows that under these conditions $F(\omega)$ can be recovered by the use of a low pass filter if $\omega_s \geq 4B_n$. This agrees with the results presented in Chapter III.

CHAPTER V

PROOF OF NYQUIST RATE REQUIREMENTS

Although it has been shown that a single filter can be described with which a band-limited function may be recovered if it is sampled at the Nyquist rate or above, it has not been shown whether or not a sequence of samples taken at a rate slower than the Nyquist rate can uniquely determine a function of time.

Determination (and recovery) are surely possible for some classes of functions when sampled below the Nyquist rate--for example, there are certain allowed frequencies below the Nyquist rate with which a band pass function may be sampled and recovered. However, if ω_s is any frequency below the Nyquist rate, there is no guarantee that the samples of an arbitrary function can determine the function without additional data.

If samples taken at such a rate can uniquely determine the function, then a network may exist with which the function can be reconstructed. Conversely, if such a sequence of samples does not uniquely determine a function (i.e., if one train determines more than one function of time), then there is no conceivable way to reconstruct the original function from the samples, assuming that no additional data are given. That sampling with impulses below the Nyquist rate does not always determine the function uniquely can be shown to be true in the following manner:

If an arbitrary band-limited function $f_1(t)$ (Figure 6a) is sampled with impulses below the Nyquist rate (ω_s less than $4\pi B_n$), a

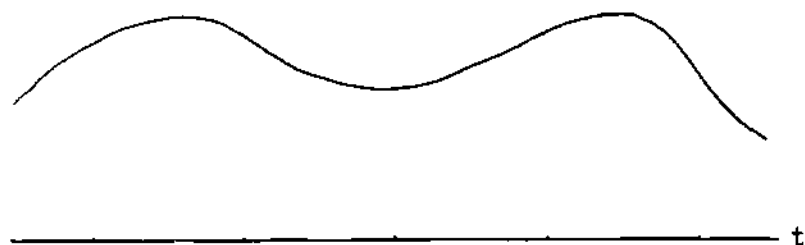


Figure 6a. $f_1(t)$ --Band-Limited Function

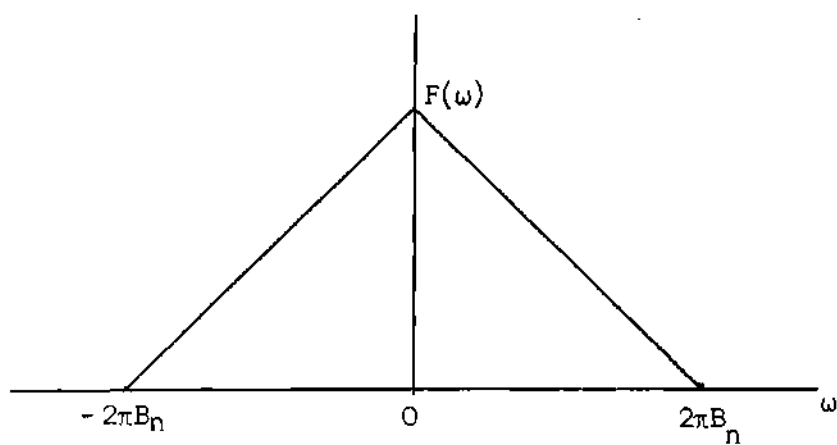


Figure 6b. Spectrum of $f_1(t)$

train of samples (Figure 7a) is generated at the rate ω_s . If this train, whose frequency spectrum is shown in Figure 7b, is applied to an ideal low pass filter of bandwidth $\omega_s/2$, the output $f_2(t)$ (Figure 8a) is also a band-limited function but with maximum frequency $\omega_s/2$. One of the properties of a low pass filter (Chapter III) of maximum frequency $\omega_s/2$ is that the response at time t_0 to a train of impulses generated at a rate ω_s impulses per second, one of which occurs at t_0 , depends only on that particular impulse. The output is equal to the value of the impulse at the instant t_0 .

If, therefore, the train of impulse samples supposedly representing $f_1(t)$ is applied to such a filter, the output $f_2(t)$ is equal to $f_1(t)$ at the sample points. However, the $f_2(t)$ cannot be equal to $f_1(t)$ for all times because $f_2(t)$ has a maximum frequency $\omega_s/2$ rps (see Figure 8b) and $f_1(t)$ has a maximum frequency $2\pi B_n$ which is greater than $\omega_s/2$.

Thus any $f(t)$ sampled below the Nyquist rate cannot always be uniquely determined since there is at least one other band-limited function with maximum frequency B_n cps having the same samples.*

*This proof was suggested to the author by Professor D. L. Finn of the Electrical Engineering Department of the Georgia Institute of Technology.



Figure 7a. $f_{1s}(t)$ --Samples of $f_1(t)$ taken below the Nyquist Rate

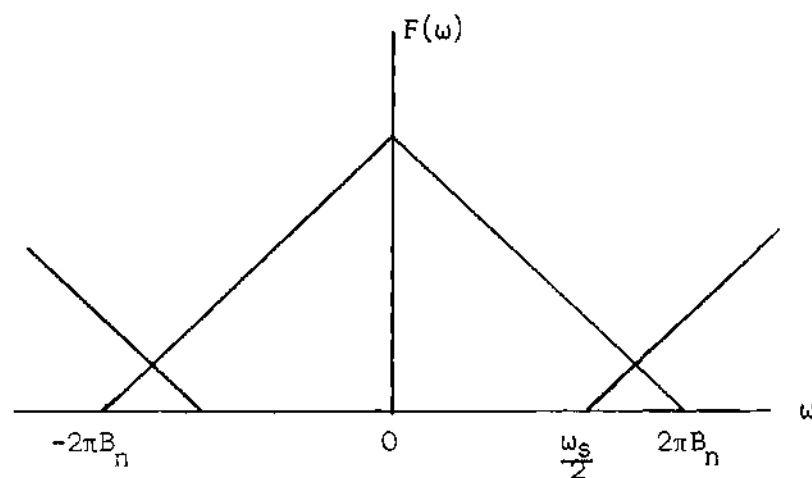


Figure 7b. Spectrum of $f_{1s}(t)$

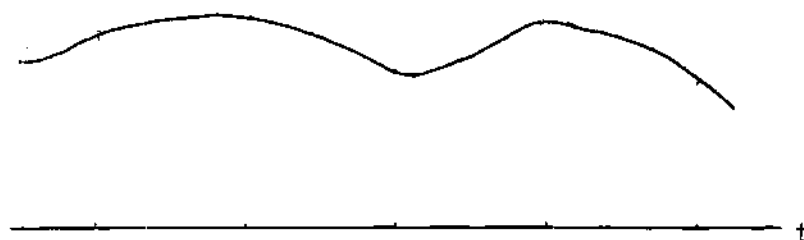


Figure 8a. $f_2(t)$ --Output of Low Pass Filter
Having $f_1(t)$ as Input

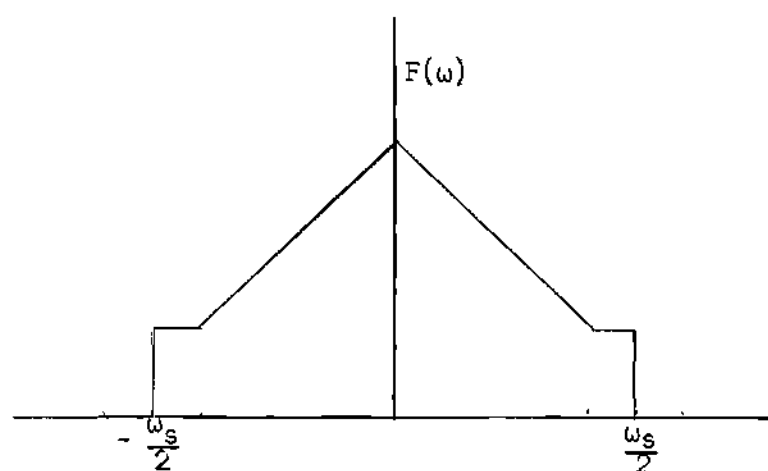


Figure 8b. Spectrum of $f_2(t)$

CHAPTER VI

SAMPLING OF BAND PASS FUNCTIONS

6.1 Definition and Sampling Constraints--A band-limited low pass function $f(t)$ was defined in Chapter II, Section 3, as one for which $F(\omega)$ was zero for all $|\omega|$ greater than $2\pi B$ rps, and for which no positive number $2\pi X$ (X less than B) existed such that $F(\omega)$ is zero for all $|\omega|$ less than $2\pi X$. If the latter restriction is changed so that there is a non-zero $2\pi X$ below which $F(\omega)$ is zero, the function can be defined as a bandpass function. Such a function is shown in Figure 9.

If the fact that $F(\omega)$ is zero in the interval $|\omega|$ less than X is neglected, the Nyquist sampling rate is $4\pi B_n$ rps. Inspection of the spectrum of $f(t)$ sampled above this Nyquist rate (see Chapter I) shows that $F(\omega)$ is indeed recoverable with a single low pass filter under these conditions. The spectrum is shown in Figure 10.

However, there are values of ω_s less than $4\pi B_n$ rps for which $F(\omega)$ is completely recoverable. Any ω_s is adequate for which there are no overlapping spectra.¹⁶ To find the requirements on ω_s , examine a section of the spectrum of $F_s(\omega)$ (Figure 11), where ω_s is an allowed frequency less than $4\pi B_n$ rps.

That part of $F_s(\omega)$ between $n\omega_s - 2\pi B$ and $n\omega_s - 2\pi X$ is the portion of $F(\omega)$ from the negative frequency spectrum that is centered on the n^{th} spectral component of the sampling function (see Chapter II, Section 4). That part of $F_s(\omega)$ between $(n+1)\omega_s - 2\pi B$ and

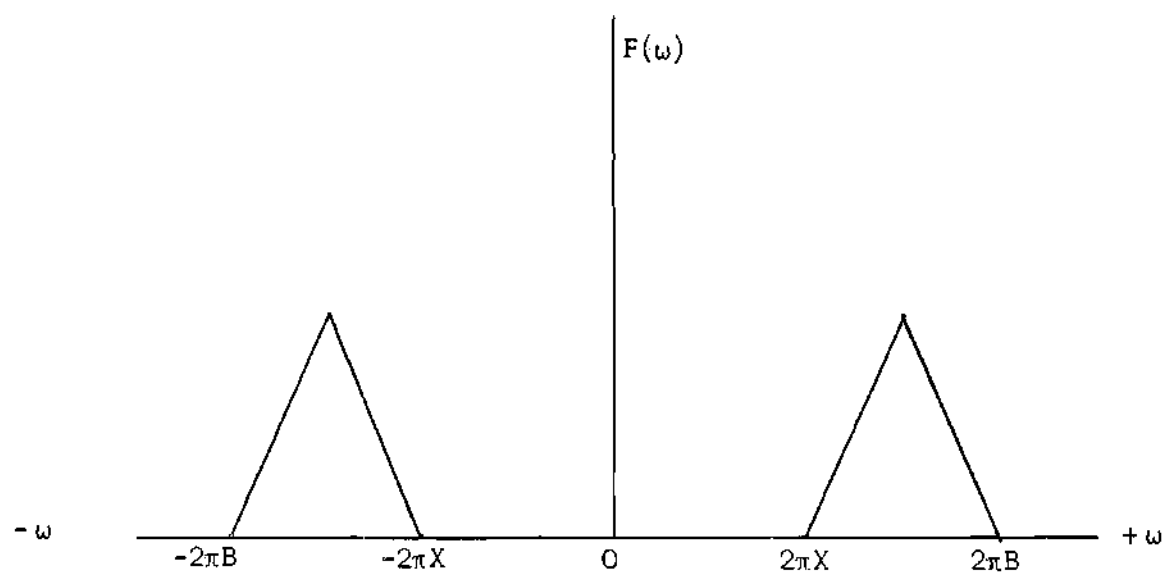


Figure 9. Band Pass Function

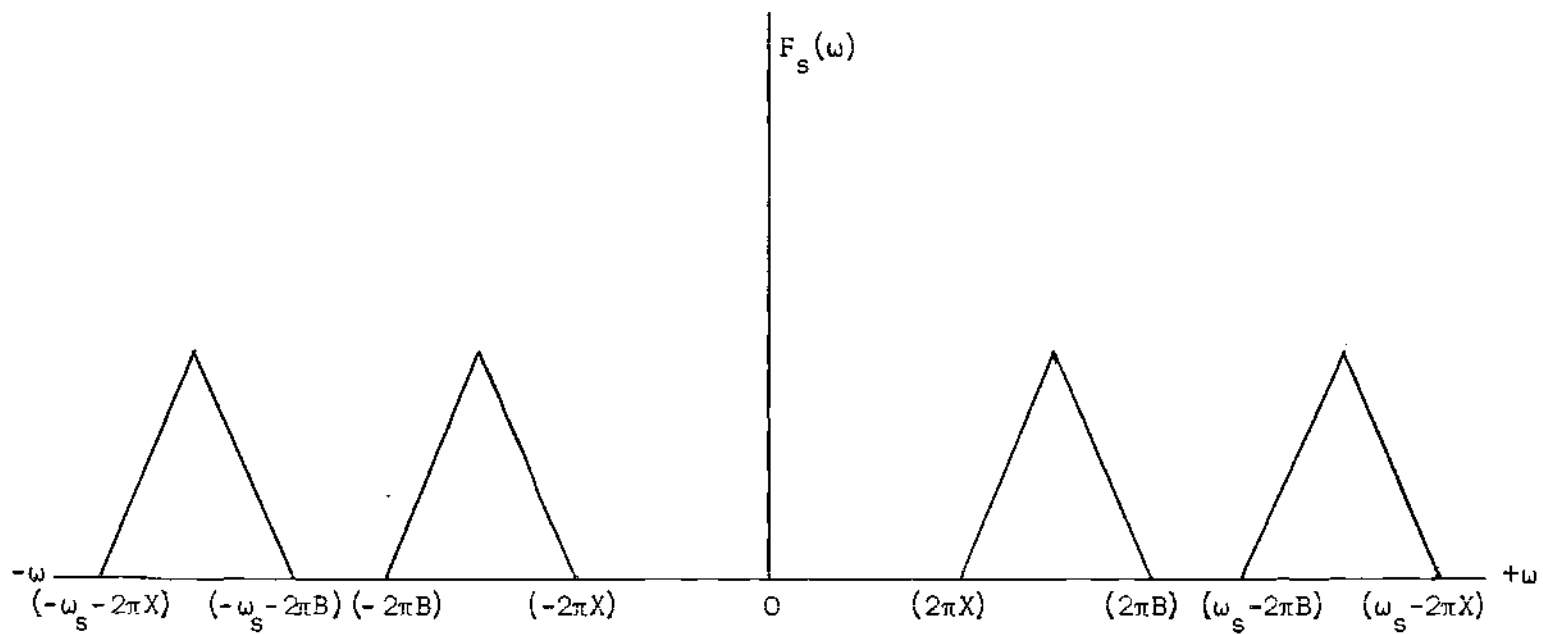


Figure 10. Spectrum of Band-Pass $f(t)$ Sampled Above the Nyquist Rate

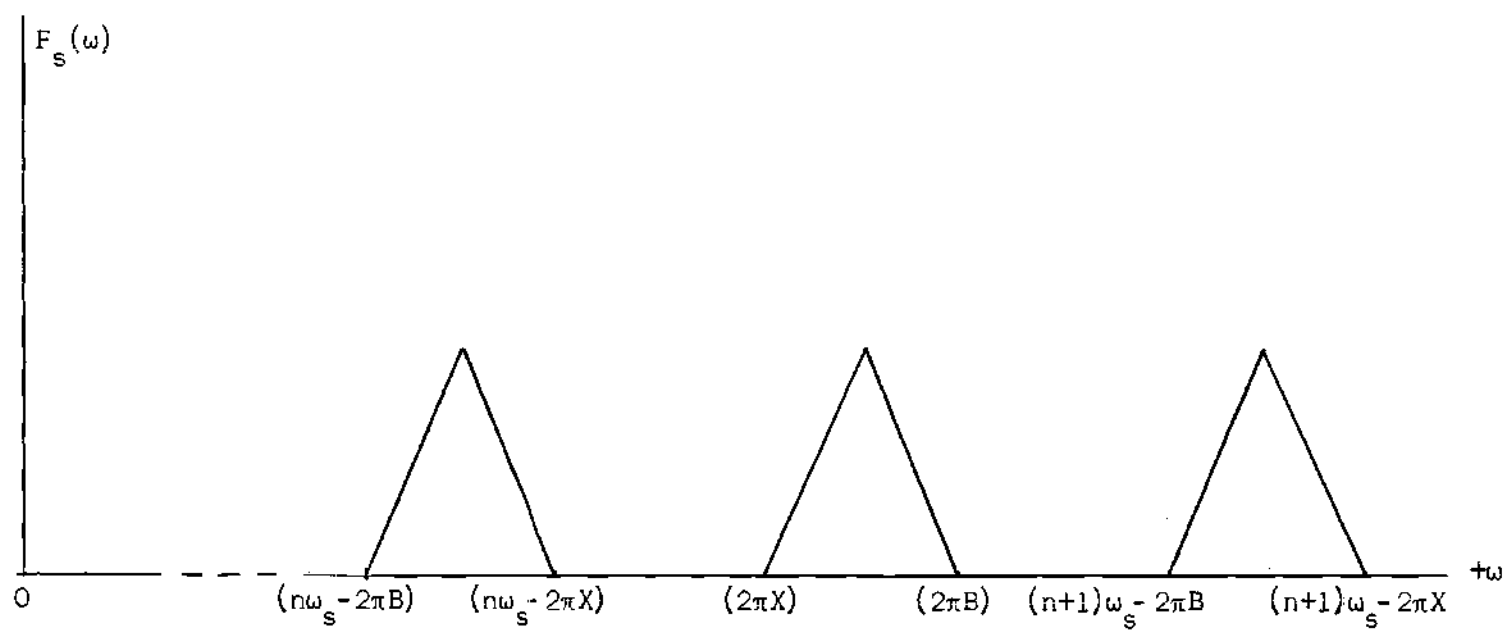


Figure 11. Section of Spectrum of Band Pass $f(t)$ Sampled Below the Nyquist Rate

$(n + 1)\omega_s - 2\pi X$ is the portion of $F(\omega)$ from the negative frequency spectrum centered on the $(n+1)^{st}$ spectral line of the sampling function. $F_s(\omega)$ between $2\pi X$ and $2\pi B_n$ is the original signal. It can be considered as the part centered on the d. c. line of the spectrum of the sampling function.

From Figure 11, the constraints are

$$n\omega_s - 2\pi X \leq 2\pi X \quad (32)$$

and

$$(n + 1)\omega_s - 2\pi B \geq 2\pi B$$

to prevent overlapping spectra. These can be rearranged:

$$n\omega_s \leq 4\pi X \quad (33)$$

$$(n + 1)\omega_s \geq 4\pi B .$$

If both sides of each inequality are divided by $2\pi X$,

$$n \frac{\omega_s}{2\pi X} \leq 2 \quad (34a)$$

$$(n + 1) \frac{\omega_s}{2\pi X} \geq 2 \frac{B}{X} . \quad (34b)$$

These last two inequalities can be plotted (assuming equality) with $\omega_s/2\pi X$ as the independent variable and n the independent variable.* The graph of (34a), shown in Figures 12 and 13, is a hyperbola. The graph

*The author is indebted to Professor D. L. Finn for this graphical approach.

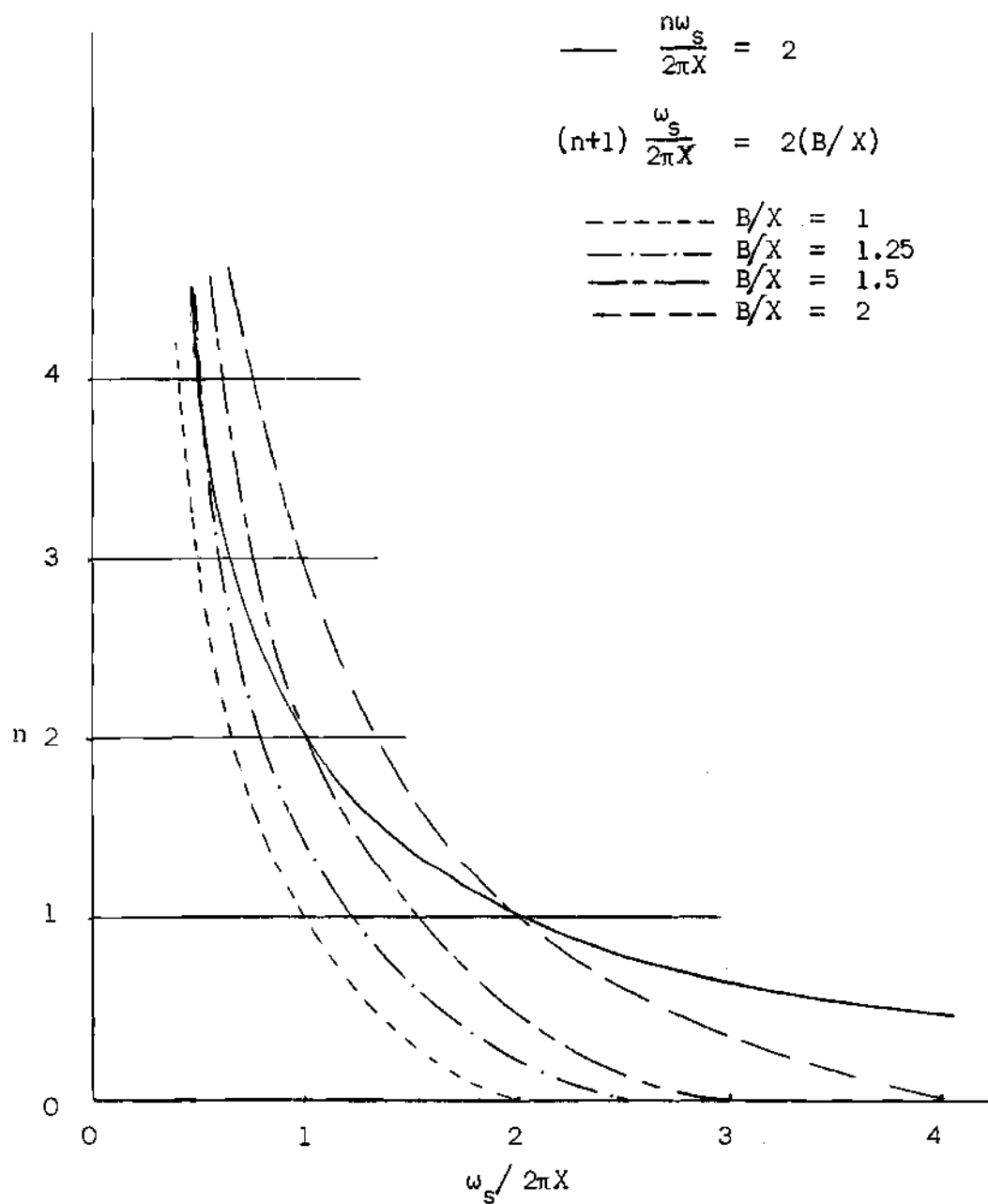


Figure 12. Constraints for Sampling of Band Pass Functions for Several Values of B/X

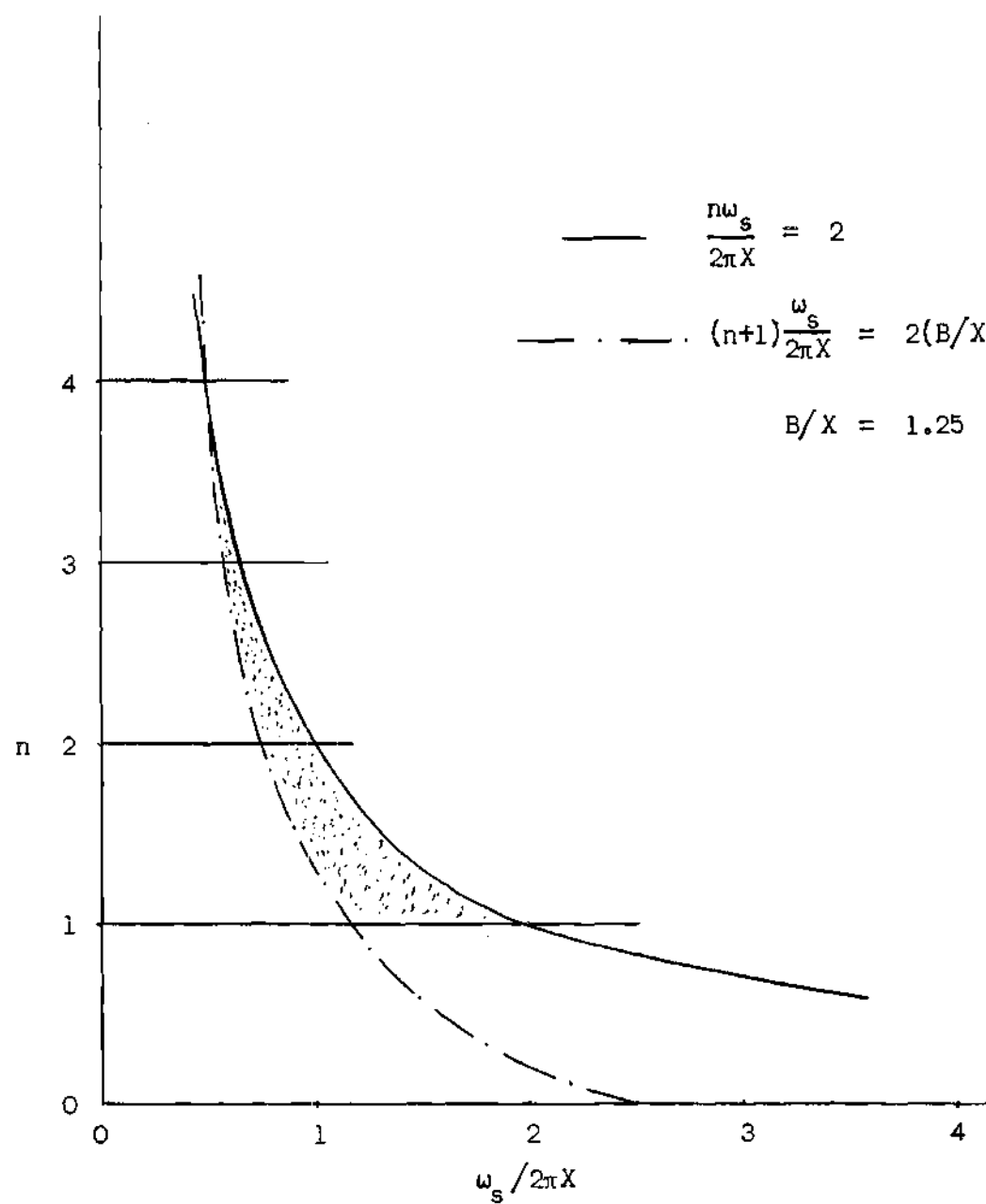


Figure 13. Constraints for Sampling of Band Pass Functions when $B/X = 1.25$

of (34b) is a family of hyperbolae having the ratio B/X as a parameter. This is shown for four values of B/X in Figure 12 and for $B/X = 1.25$ in Figure 13.

The area allowed by (34a) is to the left of its curve. The area allowed by b is to the right of its curves. The permissible values of n and $\omega_s/2\pi X$ are in the allowed area common to both curves.

It can be seen that for $B/X = 1$ all values of $\omega_s/2\pi X$ are allowable. This is a limiting condition approached when the frequency band is very narrow compared with B . The other limiting case occurs when $B/X = 2$. This would allow only one replica of $F(\omega)$ to "slip in" between these two limits. For example, let a particular $B/X = 1.25$. This is shown on Figure 12, and again by itself on Figure 13.

The curve for $B/X = 1.25$ is dashed in Figure 13. It can be seen that the minimum $\omega_s/2\pi X$ is $1/2$, and occurs when $n = 4$. This means that $\omega_s = \frac{1}{2}(2\pi X)$ is the lowest allowable sampling frequency, and the replica of $F(\omega)$ centered on the fourth harmonic of the sampling frequency will be immediately below the original $F(\omega)$. (See Figure 11.) Other possible values for $\omega_s/2\pi X$ are between 0.63 and 0.66, 0.84 and 1.0, and 1.25 and 2. Any one of these will satisfy the inequalities.

The two inequalities can also be solved analytically for a minimum ω_s in terms of X and B . If

$$n\omega_s = 4\pi X \quad (35a)$$

$$(n + 1)\omega_s = 4\pi B, \quad (35b)$$

then the ω_s satisfying these two equations is

$$\omega_s = 2(2\pi B - 2\pi X). \quad (36)$$

However, $(2\pi B - 2\pi X)$ is the bandwidth of the $f(t)$. Therefore the minimum sampling frequency ω_s must fulfill the requirements of being twice the bandwidth of $f(t)$. Also, the minimum frequency of $f(t)$, $2\pi X$, must be an integral multiple of ω_s . These calculations agree with the results obtained from Figure 13. Of course, any higher values of ω_s are permissible provided they satisfy the inequalities (33).

CHAPTER VII

CONSTRAINTS FOR SAMPLING A FUNCTION AND ITS DERIVATIVE

7.1 Introduction--If a function $f(t)$ is sampled below the Nyquist rate, it was shown that the replicas of $F(\omega)$ overlap and that recovery was not possible with a single low pass filter. It is conceivable that some means of recovery may exist under this condition if additional data are given along with the samples of the function. This section will show that sampling of the function's derivatives can supply these data under certain conditions.

7.2 Spectrum of Function Sampled Below the Nyquist Rate--If any function $g(\alpha)$ is defined by the integral

$$g(\alpha) = \int_a^z G(\alpha, \beta) d\beta, \quad (37)$$

it can be divided in the following manner:

$$g(\alpha) = \int_a^b G(\alpha, \beta) d\beta + \int_b^c G(\alpha, \beta) d\beta + \dots + \int_x^y G(\alpha, \beta) d\beta + \int_y^z G(\alpha, \beta) d\beta, \quad (38)$$
$$a < b < c < \dots < x < y < z.$$

If $f(t)$ is defined by its Fourier transform for all t :

$$f(t) = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} F(\omega') e^{j\omega' t} d\omega', \quad (39)$$

it can be similarly divided.⁴ If it is divided into $2m$ equal parts:

$$\begin{aligned}
 f(t) = \frac{1}{2\pi} \left\{ \int_{-2\pi B}^{-2\pi B(\frac{m-1}{m})} F(\omega') e^{j\omega' t} d\omega' + \int_{-2\pi B(\frac{m-1}{m})}^{-2\pi B(\frac{m-2}{m})} F(\omega') e^{j\omega' t} d\omega' + \dots \right. \\
 \left. + \int_{-\frac{2\pi B}{m}}^0 F(\omega') e^{j\omega' t} d\omega' + \int_0^{\frac{2\pi B}{m}} F(\omega') e^{j\omega' t} d\omega' + \dots \right. \\
 \left. + \int_{2\pi B(\frac{m-1}{m})}^{\frac{2\pi B}{m}} F(\omega') e^{j\omega' t} d\omega' \right\}. \quad (40)
 \end{aligned}$$

In the region $-2\pi B \leq \omega \leq 0$, a typical integral of Equation (40) is

$$\int_{2\pi B \frac{l}{m}}^{2\pi B \frac{l+1}{m}} F(\omega') e^{j\omega' t} d\omega', \quad l = -1, -2, \dots, -m. \quad (41)$$

The total expression for $f(t)$ in the negative frequency region is the sum of these integrals:

$$\int_{-2\pi B}^0 F(\omega') e^{j\omega' t} d\omega' = \sum_{l=-m}^{l=-1} \int_{2\pi B \frac{l}{m}}^{2\pi B(\frac{l+1}{m})} F(\omega') e^{j\omega' t} d\omega'. \quad (42)$$

The limits of the right-hand integral can be independent of the summation (independent of l) by applying the transformation

$$\omega = \omega' - 2\pi B l / m \quad (43)$$

to each term.

When $\omega' = 2\pi B\ell/m$, $\omega = 0$; when $\omega' = 2\pi B(\ell + 1)/m$, $\omega = 2\pi B/m$. Then under the transformation

$$\int_{-2\pi B}^0 F(\omega') e^{j\omega' t} d\omega' = \sum_{\ell=-m}^{\ell=m-1} \int_0^{\frac{2\pi B}{m}} F(\omega + 2\pi B \frac{\ell}{m}) e^{j(\omega + 2\pi B \frac{\ell}{m}) t} d\omega. \quad (44)$$

In the positive frequency region ($0 \leq \omega \leq 2\pi B$) a typical integral of Equation 40 is

$$\int_{2\pi B \frac{\ell}{m}}^{2\pi B \frac{\ell+1}{m}} F(\omega') e^{j\omega' t} d\omega', \quad \ell = 0, 1, 2, \dots, m-1. \quad (45)$$

The sum in the positive frequency region is

$$\int_0^{2\pi B} F(\omega') e^{j\omega' t} d\omega' = \sum_{\ell=0}^{\ell=m-1} \int_{2\pi B \frac{\ell}{m}}^{2\pi B \frac{\ell+1}{m}} F(\omega') e^{j\omega' t} d\omega'. \quad (46)$$

A suitable transformation of variable will again make the limits of the integral independent of the summation. Let $\omega = \omega' - 2\pi B\ell/m$; when $\omega' = 2\pi B\ell/m$, $\omega = 0$; when $\omega' = 2\pi B(\ell+1)/m$, $\omega = 2\pi B/m$. The interval of integration is again $0, 2\pi B/m$.

$$\int_0^{2\pi B} f(\omega') e^{j\omega' t} d\omega'$$

now becomes

$$\sum_{\ell=0}^{\ell=m-1} \int_0^{\frac{2\pi B}{m}} f(\omega + 2\pi B \frac{\ell}{m}) e^{j(\omega + 2\pi B \frac{\ell}{m}) t} d\omega. \quad (47)$$

The complete expression for $f(t)$ is now

$$\int_{-2\pi B}^{2\pi B} F(\omega') e^{j\omega' t} d\omega' = \sum_{\ell=-m}^{\ell=m-1} \int_0^{\frac{2\pi B}{m}} F(\omega + 2\pi B \frac{\ell}{m}) e^{j(\omega + 2\pi B \frac{\ell}{m}) t} d\omega \quad (48)$$

$$+ \sum_{\ell=0}^{\ell=m-1} \int_0^{\frac{2\pi B}{m}} F(\omega + 2\pi B \frac{\ell}{m}) e^{j(\omega + 2\pi B \frac{\ell}{m}) t} d\omega.$$

Equation (48) can be made more compact:

$$f(t) = \frac{1}{2\pi} \sum_{\ell=-m}^{\ell=m-1} \int_0^{\frac{2\pi B}{m}} F(\omega + 2\pi B \frac{\ell}{m}) e^{j(\omega + 2\pi B \frac{\ell}{m}) t} d\omega. \quad (49)$$

There are $2m$ terms. " m " is an integer greater than zero and will be constant for any particular problem.

The integrand of Equation 49 can be rearranged.

$$F(\omega + 2\pi B \frac{\ell}{m}) e^{j(\omega + 2\pi B \frac{\ell}{m}) t} = e^{j2\pi B \frac{\ell}{m} t} F(\omega + 2\pi B \frac{\ell}{m}) e^{j\omega t}. \quad (50)$$

Since the summation of Equation (49) is finite, the operations of integration and summation are interchangeable. Hence

$$f(t) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \sum_{\ell=-m}^{\ell=m-1} e^{j2\pi B \frac{\ell}{m} t} F(\omega + 2\pi B \frac{\ell}{m}) e^{j\omega t} d\omega. \quad (51)$$

If a suitable t is chosen, $t = t_0$, the integral can be represented as a type of Fourier integral. If t_0 is chosen so that $Bt_0/m = n$,

where $n = 0, \pm 1, \pm 2, \dots$, then $t_0 = nm/B$. Therefore

$$e^{j2\pi B \ell t_0/m} = e^{j2\pi n \ell} = 1,$$

and

$$f(t_0) = f\left(\frac{nm}{B}\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \sum_{\ell=-m}^{\ell=m-1} F(\omega + 2\pi B \frac{\ell}{m}) e^{j\omega \frac{nm}{B}} d\omega. \quad (52)$$

Let

$$G(\omega) = \sum_{\ell=-m}^{\ell=m-1} F(\omega + 2\pi B \frac{\ell}{m}), \quad 0 \leq \omega \leq \frac{2\pi B}{m}. \quad (53)$$

Then

$$f\left(\frac{nm}{B}\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} G(\omega) e^{j\omega \frac{nm}{B}} d\omega. \quad (54)$$

As n varies over all integral values, $f(nm/B)$ represents a train of instantaneous samples of $f(t)$ taken at intervals of m/B seconds. $G(\omega)$ can be interpreted as the Fourier transform (in the interval $0 \leq \omega \leq 2\pi B/m$) of a train of impulses, the n^{th} impulse having a value $f(nm/B)$. It is therefore the frequency spectrum of the sample train in this region, according to Equation (31), Chapter IV.¹⁷ The summation is only over a finite range of ℓ in Equation (53) since there are only a finite number of replicas in the limited frequency range of this equation.

It was shown in Chapter IV that the spectrum of a sampled function had replicas of $F(\omega)$ extending from $\omega = -\infty$ to $\omega = +\infty$. If the sampling was done with impulses, the spectrum was assumed to be periodic

with period ω_s and all of the information needed to determine $F(\omega)$ is available in any interval $m\omega_s \leq \omega \leq (m+1)\omega_s$, if the sampling is done above the Nyquist rate. This is the reason a band pass filter and some demodulation technique might be feasible for recovery as well as the usual low pass filter. This was discussed in Chapter II.

If the sampling is done at a frequency $\omega_s = 2\pi B/m$, which is below the Nyquist rate, overlapping spectra make recovery impossible without additional data, and more sophisticated techniques. However, just as in the case of sampling above the Nyquist rate, the spectrum is periodic with period ω_s and knowledge of $F_s(\omega)$ in any interval of length ω_s is sufficient to determine $F_s(\omega)$ everywhere. Therefore, $F(\omega)$ in the interval $0 \leq \omega \leq 2\pi B/m$ in Equation (53) contains all the information needed to determine the sample train $f(nm/B)$, and can be considered the frequency spectrum of a function sampled at m/B second intervals in the region. The following illustration may clarify this.

If m is chosen to be one, the sampling frequency is $2\pi B$ rps. (The Nyquist rate is $4\pi B$ rps.) It was stated in Chapter IV that there is a replica of $F(\omega)$ centered on each component of the spectrum of the impulse sampling function. For $F(\omega)$ shown in Figure 14, $F_s(\omega)$ which represents $F(\omega)$ sampled at a rate ω_s , is shown in Figure 15, in which ω_s is $2\pi B$ rps. In Figure 15, each designated component "1" and "2" can be seen to have a counterpart in the expression of $G(\omega)$ when $m = 1$, in the frequency region $0 \leq \omega \leq \omega_s$.

$$G(\omega) = F(\omega - 2\pi B) + F(\omega), \quad 0 \leq \omega \leq 2\pi B, \quad (55)$$

$$m = 1.$$

$F(\omega)$ corresponds to "1," and $F(\omega - 2\pi B)$ corresponds to "2."

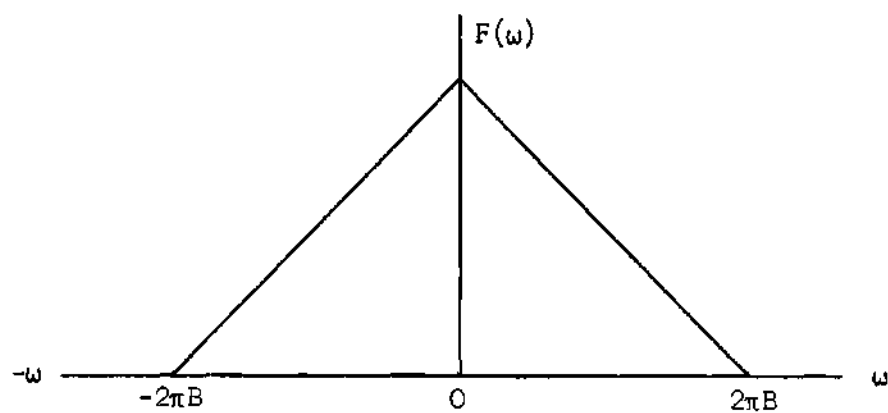


Figure 14. Unsamped $F(\omega)$

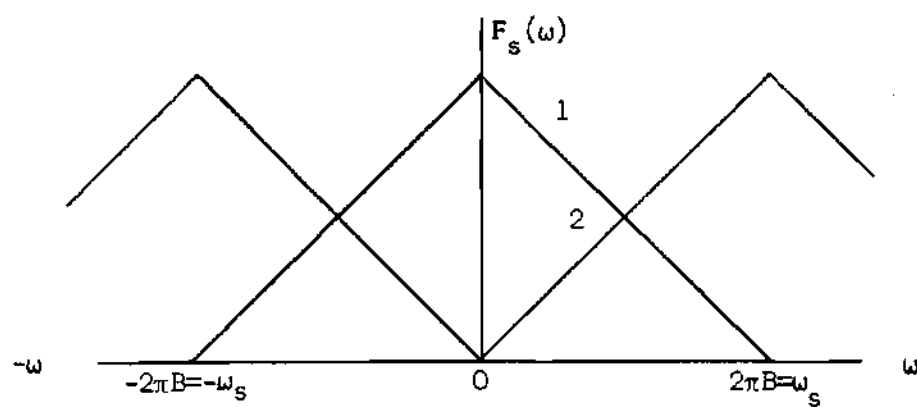


Figure 15. $F_s(\omega)$ -- Spectrum of $f(t)$ Sampled at Rate $\omega_s = 2\pi B$

7.3 Determination of a Function from Derivatives--If, as has been assumed,

$$f(t) = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} F(\omega) e^{j\omega t} d\omega, \quad (56)$$

for an appropriate class of functions, it is true that

$$\frac{d^k f(t)}{(dt)^k} = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} (j\omega)^k F(\omega) e^{j\omega t} d\omega. \quad (57)$$

Also, if $F(\omega) = H_1(\omega) + H_2(\omega)$,

$$f(t) = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} [H_1(\omega) e^{j\omega t} + H_2(\omega) e^{j\omega t}] d\omega, \quad (58)$$

and

$$\frac{d^k f(t)}{(dt)^k} = \frac{1}{2\pi} \int_{-2\pi B}^{+2\pi B} [(j\omega)^k H_1(\omega) e^{j\omega t} + (j\omega)^k H_2(\omega) e^{j\omega t}] d\omega. \quad (59)$$

The k^{th} derivative of $f(t)$ evaluated at $t = nm/B$, denoted $f^k(nm/B)$ is, with reference to Equation (52)

$$f^k\left(\frac{nm}{B}\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \sum_{\ell=-m}^{\ell=m-1} \{[j(\omega + 2\pi B \frac{\ell}{m})]^k F(\omega + 2\pi B \frac{\ell}{m})\} e^{j\omega(\frac{nm}{B})} d\omega. \quad (60)$$

The set of derivatives $f^0(nm/B, f^1(nm/B, \dots, f^k(nm/B)$ can be developed. If

$$G^k(\omega) = \sum_{l=-m}^{l=m-1} [j(\omega + 2\pi B \frac{l}{m})]^k F(\omega + 2\pi B \frac{l}{m}), \quad 0 \leq \omega \leq \frac{2\pi B}{m}, \quad (61)$$

then

$$f^0(\frac{nm}{B}) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} G^0(\omega) e^{j\omega \frac{nm}{B}} d\omega, \quad (62)$$

$$f^1(\frac{nm}{B}) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} G^1(\omega) e^{j\omega \frac{nm}{B}} d\omega, \quad (63)$$

$$\vdots$$

$$f^k(\frac{nm}{B}) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} G^k(\omega) e^{j\omega \frac{nm}{B}} d\omega. \quad (64)$$

There are $2m$ terms of the form $[j(\omega + 2\pi B l/m)]^k F(\omega + 2\pi B l/m)$ in each $G^k(\omega)$. In a sampling problem the $f^k(nm/B)$, and hence the $G^k(\omega)$ are known, but the $F(\omega + 2\pi B l/m)$ are unknown. For any given ω there are $2m$ of these unknowns. The $2m$ equations necessary to solve for these unknowns may be obtained by letting $k = 2m - 1$. That is, by obtaining samples of the function and $2m - 1$ derivatives.

These equations are:

$$f\left(\frac{nm}{B}\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \left\{ F(\omega - 2\pi B) + \dots + F(\omega) + \dots + F\left(\omega + 2\pi B\left(\frac{m-1}{m}\right)\right) \right\} e^{j\omega \frac{nm}{B}} d\omega \quad (65)$$

•
•
•

$$f^k\left(\frac{nm}{B}\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \left\{ [j(\omega - 2\pi B)]^k F(\omega - 2\pi B) + \dots + (j\omega)^k F(\omega) + \dots + [j(\omega + 2\pi B\left(\frac{m-1}{m}\right))]^k F(\omega + 2\pi B\left(\frac{m-1}{m}\right)) \right\} e^{j\omega \frac{nm}{B}} d\omega$$

•
•
•

$$f^{2m-1}\left(\frac{nm}{B}\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \left\{ [j(\omega - 2\pi B)]^{2m-1} F(\omega - 2\pi B) + \dots + (j\omega)^{2m-1} F(\omega) + \dots + [j(\omega + 2\pi B\left(\frac{m-1}{m}\right))]^{2m-1} F(\omega + 2\pi B\left(\frac{m-1}{m}\right)) \right\} d\omega$$

k again denotes a general term and will continue to do so.

From this set of equations it is possible to find any $F(\omega + 2\pi B\ell/m)$, assuming that each is independent.⁴ These unknowns are independent because $F(\omega)$ is assumed to be an arbitrary band-limited function. The determination of all unknowns $F(\omega + 2\pi B\ell/m)$ is sufficient to determine $f(\omega)$ over the entire range $0 \leq \omega \leq 2\pi B$.

It has been shown that $f(nm/B)$ can be considered a set of instantaneous samples of $f(t)$ taken at intervals m/B seconds apart. It will be shown that if $f(t)$ and its $2m - 1$ time derivatives are sampled at m/B second intervals, the set of equations can be set up from which $F(\omega)$ can be recovered.

This, then, is a sufficient constraint for sampling a function and its derivatives:

A function $f(t)$, band-limited to B cps, is completely determined if instantaneous samples of $f(t)$ and $2m - 1$ derivatives of $f(t)$ are taken at intervals m/B seconds apart.

7.4 Recovery of the Function--One method of reconstructing the original $f(t)$ can be developed in the following manner:

Let $G^0(\omega)$, $G^1(\omega)$, ..., $G^{2m-1}(\omega)$ be defined by Equation (61). Assume some functions $I_0(\omega)$, $I_1(\omega)$, ..., $I_k(\omega)$, ..., $I_{2m-1}(\omega)$ exist such that

$$G^0(\omega)I_0(\omega) + G^1(\omega)I_1(\omega) + \dots + G^k(\omega)I_k(\omega) + \dots + G^{2m-1}(\omega)I_{2m-1}(\omega) = F(\omega), \quad (66)$$

$$0 \leq \omega \leq \frac{2\pi B}{m}.$$

If the $I_k(\omega)$ are realizable filters, recovery of $F(\omega)$ in the range of Equation (56) is the process of sending each train of impulse

samples through the appropriate sets of filter and summing the outputs to get the desired function.¹⁸ The process is shown in Figure 16. The low pass filters are necessary because $G(\omega)$ has been defined to be zero for all ω greater than $2\pi B/m$, and all $\omega < 0$, and the spectrum of the train of samples extends from $\omega = -\infty$ to $\omega = +\infty$. This type of filter is not physically realizable.

The requirements for the filters $I_k(\omega)$ can be developed by multiplying each side of Equation (61) by the appropriate $I_k(\omega)$.

If

$$G^0(\omega) I_0(\omega) = \sum_{\ell=-m}^{\ell=m-1} F(\omega + 2\pi B \frac{\ell}{m}) I_0(\omega), \quad (67)$$

and

$$G^1(\omega) I_1(\omega) = \sum_{\ell=-m}^{\ell=m-1} [j(\omega + 2\pi B \frac{\ell}{m})] F(\omega + 2\pi B \frac{\ell}{m}) I_1(\omega) \quad (68)$$

$$\begin{array}{c} \circ \\ \circ \\ \circ \end{array} \quad \begin{array}{c} \circ \\ \circ \\ \circ \end{array}$$

$$G^k(\omega) I_k(\omega) = \sum_{\ell=-m}^{\ell=m-1} [j(\omega + 2\pi B \frac{\ell}{m})]^k F(\omega + 2\pi B \frac{\ell}{m}) I_k(\omega)$$

$$\begin{array}{c} \circ \\ \circ \\ \circ \end{array} \quad \begin{array}{c} \circ \\ \circ \\ \circ \end{array}$$

$$G^{2m-1} I_{2m-1}(\omega) = \sum_{\ell=-m}^{\ell=m-1} [j(\omega + 2\pi B \frac{\ell}{m})]^{2m-1} F(\omega + 2\pi B \frac{\ell}{m}) I_{2m-1}(\omega),$$

then the requirement of Equation (66) can be expressed in terms of the individual $F(\omega + 2\pi B \ell/m)$ by adding the right-hand sides of Equation (67).

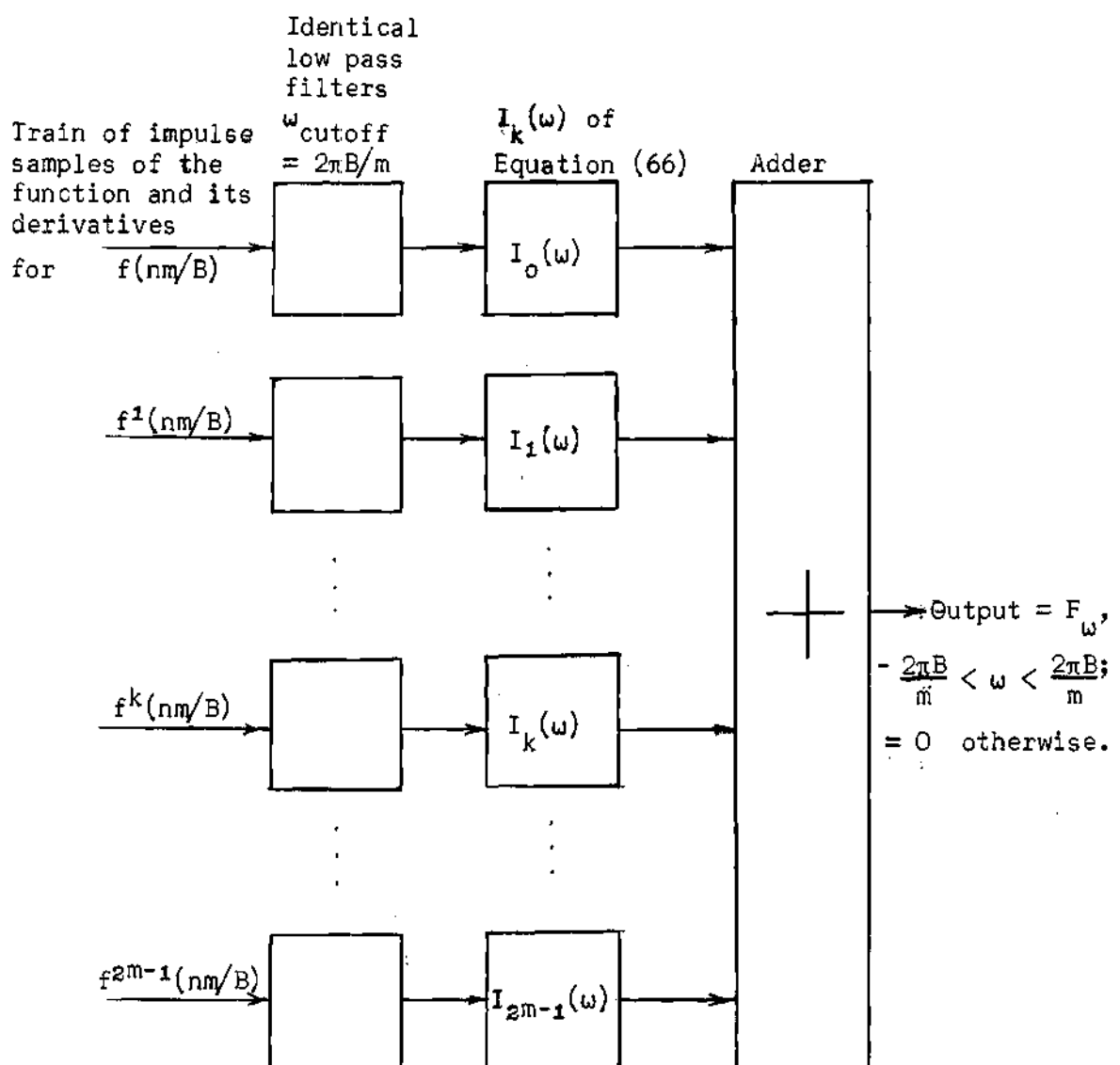


Figure 16. Method of Partial Recovery of $F(\omega)$ as Indicated by Equation (66)

$$\sum_{\ell=-m}^{\ell=m-1} F(\omega+2\pi B \frac{\ell}{m}) I_0(\omega) + \sum_{\ell=-m}^{\ell=m-1} [j(\omega+2\pi B \frac{\ell}{m})] F(\omega+2\pi B \frac{\ell}{m}) I_1(\omega) + \dots \quad (69)$$

$$+ \sum_{\ell=-m}^{\ell=m-1} [j(\omega+2\pi B \frac{\ell}{m})]^k F(\omega+2\pi B \frac{\ell}{m}) I_k(\omega) + \dots$$

$$+ \sum_{\ell=-m}^{\ell=m-1} [j(\omega+2\pi B \frac{\ell}{m})]^{2m-1} F(\omega+2\pi B \frac{\ell}{m}) I_{2m-1} = F(\omega) .$$

When all terms are combined under one summation symbol:

$$F(\omega) = \sum_{\ell=-m}^{\ell=m-1} F(\omega+2\pi B \frac{\ell}{m}) I_0(\omega) + \dots + [j(\omega+2\pi B \frac{\ell}{m})]^{2m-1} F(\omega+2\pi B \frac{\ell}{m}) I_{2m-1}(\omega) . \quad (70)$$

The terms in Equation (70) can be rearranged.

$$F(\omega-2\pi B) \{I_0(\omega) + [j(\omega-2\pi B)] I_1(\omega) + \dots + [j(\omega-2\pi B)]^{2m-1} I_{2m-1}(\omega)\} + \dots \quad (71)$$

$$+ F(\omega) \{I_0(\omega) + [j\omega] I_1(\omega) + \dots + [j\omega]^{2m-1} I_{2m-1}(\omega)\} + \dots$$

$$+ F(\omega+2\pi B \frac{\ell}{m}) \{I_0(\omega) + [j(\omega+2\pi B \frac{\ell}{m})] I_1(\omega) + \dots$$

$$+ [j(\omega+2\pi B \frac{\ell}{m})]^{2m-1} I_{2m-1}(\omega)\} + \dots + F(\omega-2\pi B(\frac{m-1}{m}))$$

$$\times \{I_0(\omega) + [j(\omega+2\pi B(\frac{m-1}{m}))] I_1(\omega) + \dots + [j(\omega+2\pi B(\frac{m-1}{m}))]^{2m-1}$$

$$\times I_{2m-1}(\omega)\} = F(\omega) .$$

Since it is required that this sum be equal to $F(\omega)$ at all ω , $0 \leq \omega \leq 2\pi B/m$, for an arbitrary band-limited $F(\omega)$, all the coefficients of $F(\omega + 2\pi B\ell/m)$ must be zero, except the coefficient of $F(\omega)$ which must be one. Each coefficient, however, is a sum with $2m$ terms, since the $I_k(\omega)$ are summed from $k = 0$ to $k = 2m - 1$. There are, in any particular problem, $2m$ coefficients, and another set of equations is thus developed. This set contains the $I_k(\omega)$ as the unknowns to be found. Expressed in a compact form, this set is

$$\sum_{\ell=-m}^{\ell=m-1} \sum_{k=0}^{k=2m-1} I_k(\omega) [j(\omega + 2\pi B\frac{\ell}{m})]^k = 0, \quad \ell \neq 0 \quad (72)$$

$$= 1, \quad \ell = 0.$$

In the case where a unique solution of the $I_k(\omega)$ of Equation (66) exists, solution of these Equations (72) gives these $I_k(\omega)$ directly. If the response to an impulse is desired, the inverse Fourier transforms of the $I_k(\omega)$ can be found.

This method of evaluating an $I_k(\omega)$ is valid only in the frequency range $0 \leq \omega \leq 2\pi B/m$. In order to determine $I_k(\omega)$ for other frequency ranges it is possible to return to the operation transforming

$$f(t) = \int_{-2\pi B}^{+2\pi B} F(\omega') e^{j\omega' t} d\omega' \quad (73)$$

to

$$f(t) = \int_0^{\frac{2\pi B}{m}} \sum_{\ell=-m}^{\ell=m-1} G(\omega', t) e^{j\omega' t} d\omega' \quad (74)$$

and, instead, transform $f(t)$ to

$$f(t) = \int_{\frac{2\pi Br}{m}}^{\frac{2\pi B(r+1)}{m}} \sum_{\ell=-m}^{\ell=m-1} G(\omega', t) e^{j\omega' t} d\omega', \quad (75)$$

where r is an integer with value greater than zero and less than m .

If m is larger than one, it will be necessary to make the transformation of Equation (73) for each frequency region ($r = 0, 1, 2, \dots, m-1$) and evaluate the $I_k(\omega)$ for that region. In general there are m positive ranges having $2m$ components each, since each $F(\omega + 2\pi B\ell/m)$ is a separate component. Therefore $2m^2$ filters in general are required to reconstruct the function.

An alternative possibility is to procure $I_k(\omega)$ in Equations (72) so that all $F(\omega + 2\pi B\ell/m)$ of Equation (71) may be recovered, and then transform these $F(\omega + 2\pi B\ell/m)$ to their appropriate place in the original spectrum of $F(\omega)$ by some means. The number of filters $I_k(\omega)$ required is still $2m^2$ for a real $f(t)$, but the mechanics of finding the $I_k(\omega)$ is much simpler since no more transformations are required. The only difference in the technique of solving for $F(\omega + 2\pi B\ell/m)$ when ℓ is not zero is that Equation (72) is set equal to unity for the new ℓ and zero for all other values of ℓ .

It is not necessary to go through these two procedures to recover $F(\omega)$ for negative frequencies because if $I_k(\omega)$ is defined to be $I_k^*(\omega)$ (conjugate), and if $F(\omega)$ is recovered in the interval $0 \leq \omega \leq 2\pi B/m$, it will also be determined in the equivalent negative frequency region $0 \geq \omega \geq -2\pi B/m$. This can be proved in the following manner:

In the original $F(\omega)$, $F(-\omega) = F^*(\omega)$; also $G_k(-\omega) = G_k^*(\omega)$, since all time functions involved in the sampling operation are real. The operation of taking a conjugate is distributive, that is, if A and B are two complex functions, and if

$$A + B = C, \text{ and}$$

$$AB = D,$$

then

$$A^* + B^* = C^*, \tag{76}$$

$$A^*B^* = D^*.$$

Equation (66) can be expressed in the following compact form for convenience:

$$\sum_{k=0}^{2m-1} G_k(\omega) I_k(\omega) = F(\omega). \tag{77}$$

It follows that

$$\sum_{k=0}^{2m-1} G_k(-\omega) I_k(-\omega) = F(-\omega). \tag{78}$$

Now, $G_k^*(\omega) = G_k(-\omega)$. If $I_k^*(\omega) = I_k(-\omega)$, then by Equations (76), (79) is true:

$$\sum_{k=0}^{2m-1} G_k^*(\omega) I_k^*(\omega) = F^*(\omega). \tag{79}$$

But $F^*(\omega) = F(-\omega)$ in the original function, and recovery is thus effected for negative frequency. The requirement that $I_k^*(\omega)$ be equal

to $I_k(-\omega)$ is met by all functions that can be approximated by a realizable network.

So, in any physical case, it is only needed to determine $F(\omega)$ for positive frequency. The corresponding negative frequency is taken care of correctly because of the definition of $I_k(\omega)$.

Example:

It is assumed that a function is sampled at $1/B$ second intervals. This is half the Nyquist rate. All necessary derivatives are available. What derivatives are needed and what filters $(I_k(\omega))$ are necessary to reconstruct the function?

According to Section 3 of this chapter, $2m-1$ derivatives are required if intervals are m/B seconds apart. For this example $m = 1$, so samples of the function and its first derivative are sufficient for recovery. Then, $f^0(n/B)$ and $f^1(n/B)$ are, by Equations (65)

$$f\left(\frac{n}{B}\right) = \frac{1}{2\pi} \int_0^{2\pi B} \{F(\omega - 2\pi B) + F(\omega)\} e^{j\omega \frac{n}{B}} d\omega, \quad (80)$$

$$f'\left(\frac{n}{B}\right) = \frac{1}{2\pi} \int_0^{2\pi B} \{[j(\omega - 2\pi B)] F(\omega - 2\pi B) + [j\omega] F(\omega)\} e^{j\omega \frac{n}{B}} d\omega. \quad (81)$$

$\{F(\omega - 2\pi B) + F(\omega)\}$ is to be multiplied by $I_0(\omega)$, and $\{[j(\omega - 2\pi B)] F(\omega - 2\pi B) + [j\omega] F(\omega)\}$ is to be multiplied by $I_1(\omega)$ so that

$$I_0(\omega) \{F(\omega - 2\pi B) + F(\omega)\} + I_1(\omega) \{[j\omega] F(\omega) + [j(\omega - 2\pi B)] F(\omega - 2\pi B)\} = F(\omega). \quad (82)$$

Rearranging terms

$$F(\omega) [I_0(\omega) + j\omega I_0(\omega)] + F(\omega - 2\pi B) [I_0(\omega) + j(\omega - 2\pi B) I_0(\omega)] = F(\omega) \quad (83)$$

Therefore

$$I_0(\omega) + I_1(\omega) j\omega = 1, \quad (84)$$

$$I_0(\omega) + I_1(\omega) j(\omega - 2\pi B) = 0.$$

When solved, these equations yield

$$I_0(\omega) = 1/2j\pi B = I_0^*(-\omega), \quad (85)$$

$$I_0(\omega) = (2\pi B - \omega)/2\pi B = I_1^*(-\omega), \quad 0 \leq \omega \leq 2\pi B.$$

Neither of these is physically realizable, but might be approximated by a suitable network over the restricted frequency range.

The following statement can now be made: If the samples of $f(t)$ and $f^1(t)$ are taken at a rate of B cps, and the samples of $f(t)$ are applied to the filter designated $I_0(\omega)$, and those of $f^1(t)$ are applied to the filter designated $I_1(\omega)$, after having been passed through low pass filters cutting off at B cps, then the sum of the responses of $I_0(\omega)$ and $I_1(\omega)$ to their respective inputs is exactly $F(\omega)$. The spectrum of the sample train of $f(t)$ is shown in Figure 15.

CHAPTER VIII

CLUSTER SAMPLING

8.1 Determination of the Function by the Samples.--It was shown in the preceding chapter that $f(t)$ might be reconstructed when the sampling was below the Nyquist rate. The requirements for the sampling rate, and a method of finding recovery filters needed under these conditions were developed.

In this chapter it will be shown that there is another method of getting the information needed for reconstruction of a band-limited function sampled below the Nyquist rate. This will be called "cluster" sampling.⁵

Ordinarily, if a function is sampled at intervals of m/B seconds, the n^{th} sample is considered as that one taken at nm/B seconds and the subsequent sample, denoted the $(n+1)^{\text{st}}$, is taken at $(n+1)m/B$ seconds. The interval between all samples is m/B seconds.

In the operation which shall be called cluster sampling, instantaneous samples are taken periodically but at unequal intervals. This is illustrated in Figure 17. If the sampling function is periodic with period m/B seconds, the individual samples can be denoted $f(nm/B + \alpha_1)$, $f(nm/B + \alpha_2)$, ..., $f(nm/B + \alpha_k)$, ..., $f(nm/B + \alpha_r)$. α_r is the time difference between a reference $t = nm/B$ and $t = nm/B + \alpha_k$. All α are shown positive in Figure 17.

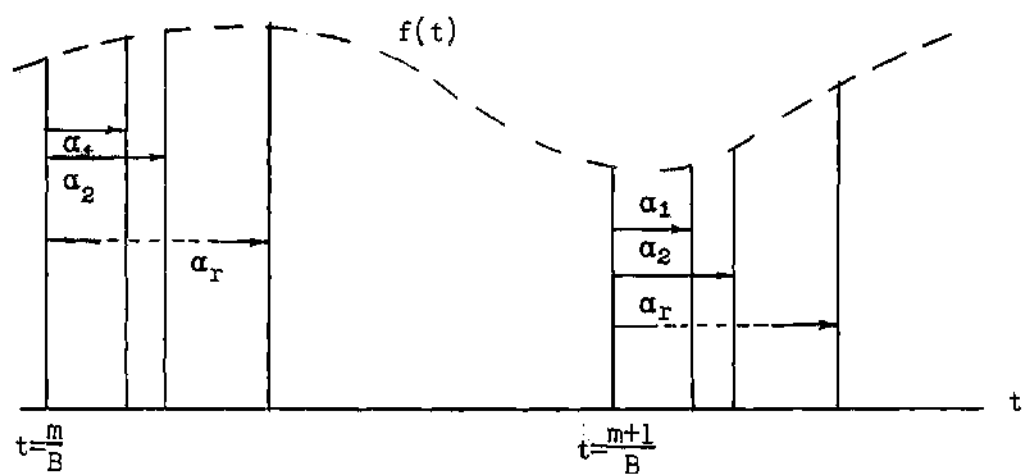


Figure 17. Example of Cluster Sampling

This operation can be analyzed by the same general technique that was used in developing the constraints for derivative sampling.

Equation (86) is an expression for $f(t)$ band-limited to B cps. This was derived in Chapter VII as Equation (51).

$$f(t) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \sum_{l=-m}^{l=m-1} e^{j(2\pi B \frac{l}{m} t)} F(\omega + 2\pi B \frac{l}{m}) e^{j\omega t} d\omega \quad (86)$$

$$f(t+\alpha) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \sum_{l=-m}^{l=m-1} e^{j2\pi B \frac{l}{m} (t+\alpha)} F(\omega + 2\pi B \frac{l}{m}) e^{j\omega (t+\alpha)} d\omega \quad (87)$$

If $t = nm/B$, where $n = 0, \pm 1, \pm 2, \dots$,

$$f(\frac{nm}{B} + \alpha) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \sum_{l=-m}^{l=m-1} e^{j[2\pi B \frac{l}{m} (\frac{nm}{B} + \alpha)]} F(\omega + 2\pi B \frac{l}{m}) e^{j\omega (\frac{nm}{B} + \alpha)} d\omega. \quad (88)$$

Since l , m , and n are all integers,

$$e^{j2\pi B \frac{lmn}{B}} = 1.$$

Then

$$f(\frac{nm}{B} + \alpha) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \sum_{l=-m}^{l=m-1} e^{j2\pi B \frac{l\alpha}{m}} F(\omega + 2\pi B \frac{l}{m}) e^{j\omega (\frac{nm}{B} + \alpha)} d\omega. \quad (89)$$

But $f(nm/B + \alpha)$ represents a train of samples of $f(t)$ taken at m/B second intervals (assuming that α is constant). If a certain train is made up of samples taken at a reference time ($\alpha = 0$), others taken at

times up to α_r , the complete collection of samples can be denoted by

$$f(nm/B) + f(nm/B + \alpha_1) + \dots + f(nm/B + \alpha_k) + f(nm/B + \alpha_r), \quad (90)$$

where n assumes all integral values from $-\infty$ to $+\infty$.

Next, assume that the samples are separated into trains of periodic, equally spaced samples. Then there is one set denoted by $f(nm/B)$, another by $f(nm/B + \alpha_1)$, and so forth. They are assumed to be separated so that they may be applied to different filters.

By Equation (87)

$$f\left(\frac{nm}{B} + \alpha_k\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \sum_{\ell=-m}^{\ell=m-1} e^{j2\pi B \alpha_k \frac{\ell}{m}} F\left(\omega + 2\pi B \frac{\ell}{m}\right) e^{j\omega\left(\frac{nm}{B} + \alpha_k\right)} d\omega \quad (91)$$

For each α_k there is a different equation. The whole array of $f(nm/B + \alpha_k)$ is shown on the following page:

$$f\left(\frac{nm}{B}\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \left\{ F(\omega - 2\pi B) + \dots + F(\omega) + \dots + F\left[\omega + 2\pi B\left(\frac{m-1}{m}\right)\right] \right\} e^{j\omega \frac{nm}{B}} d\omega \quad (91)$$

•
•
•

$$f\left(\frac{nm}{B} + \alpha_k\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \left\{ e^{-j2\pi B\alpha_k} F(\omega - 2\pi B) + \dots + F(\omega) + \dots + e^{j2\pi B\alpha_k \left(\frac{m-1}{m}\right)} F\left[\omega + 2\pi B\left(\frac{m-1}{m}\right)\right] \right\} e^{j\omega \left(\frac{nm}{B} + \alpha_k\right)} d\omega$$

•
•
•

$$f\left(\frac{nm}{B} + \alpha_r\right) = \frac{1}{2\pi} \int_0^{\frac{2\pi B}{m}} \left\{ e^{-j2\pi B\alpha_r} F(\omega - 2\pi B) + \dots + F(\omega) + \dots + e^{j2\pi B\alpha_r \left(\frac{m-1}{m}\right)} F\left[\omega + 2\pi B\left(\frac{m-1}{m}\right)\right] \right\} e^{j\omega \left(\frac{nm}{B} + \alpha_r\right)} d\omega .$$

There are $r+1$ equations.

As in the development of derivative sampling, it can be seen that a set of simultaneous equations is evolved. Since there are $2m$ terms in each equation, a unique solution possibly exists if there are $2m$ equations, that is, if $r = 2m - 1$, and if the various $F(\omega + 2\pi B\ell/m)$ of Equations (92) are independent.

The results to be proved may be stated in the following manner: If a function $f(t)$, band-limited to B cps, is sampled at a rate B/m cps, it is completely determined by its samples if, in addition to the train of samples taken at $n\pi/B$ seconds, there are also $2m - 1$ trains of samples taken at $t = n\pi/B + \alpha_k$ seconds; $k = 1, 2, 3, 2m - 1$. Also, $0 < \alpha_k < \pi/B$, and $\alpha_i \neq \alpha_j$. The average number of samples taken per second is $2B$.

$$B/m \frac{\text{samples/second}}{\text{train}} \times 2m \text{ trains} = 2B \text{ samples/second}.$$

8.2 Recovery of the Sampled Function.--Recovery of $f(t)$ and consequent proof of the above statement, can be effected by methods similar to those used to recover a function from its own and its derivative's samples.

If the integrand of Equation (91) is designated $G_k(\omega)e^{j(n\pi/B + \alpha_k)}$, where,

$$G_k(\omega) = \sum_{\ell=-m}^{\ell=m-1} e^{j2\pi B \frac{\ell}{m} \alpha_k} F(\omega + 2\pi B \frac{\ell}{m}), \quad 0 \leq \omega \leq \frac{2\pi B}{m}, \quad (93)$$

then some functions $I_k(\omega)$ may be assumed to exist such that the following Equation (94) is true.

$$I_0(\omega) G_0(\omega) + I_1(\omega) G_1(\omega) + \dots + I_k(\omega) G_k(\omega) + \dots \quad (94)$$

$$+ I_{2m-1} G_{2m-1}(\omega) = F(\omega) .$$

$G_k(\omega)$ can be interpreted similarly to the $G_k(\omega)$ for derivative sampling in Equation (61). In the interval $0 \leq \omega \leq 2\pi B/m$, $G_0(\omega)$ represents the frequency spectrum of the train of samples of the function taken at $t = nm/B$ seconds; $G_k(\omega)$ represents the spectrum of the train of samples of the function taken at $t = nm/B + \alpha_k$ seconds.

Each term $I_k(\omega) G_k(\omega)$ can be expanded by substituting the right-hand side of Equation (93) for $G_k(\omega)$.

$$I_k(\omega) G_k(\omega) = I_k(\omega) \left\{ e^{-j2\pi B \alpha_k} F(\omega - 2\pi B) + \dots + F(\omega) + \dots \right. \\ \left. + e^{j2\pi B \alpha_k \frac{1}{m}} F(\omega + 2\pi B \frac{1}{m}) + \dots + e^{j2\pi B \alpha_k (\frac{m-1}{m})} F[\omega + 2\pi B (\frac{m-1}{m})] \right\} . \quad (95)$$

If the $I_k(\omega) G_k(\omega)$ are summed [Equation (94)] and rearranged as in the preceding chapter, the following is obtained:

$$F(\omega - 2\pi B) \{ I_0(\omega) + e^{-j2\pi B \alpha_1} I_1(\omega) + e^{-j2\pi B \alpha_2} I_2(\omega) + \dots \\ + I_{2m-1}(\omega) e^{-j2\pi B \alpha_{2m-1}} \} + \dots + F(\omega) \{ I_0(\omega) + I_1(\omega) + I_2(\omega) + \dots \\ + I_{2m-1}(\omega) \} + \dots + F(\omega + 2\pi B (\frac{m-1}{m})) \{ I_0(\omega) + I_1(\omega) e^{j2\pi B (\frac{m-1}{m}) \alpha_1} + \dots \\ + \dots + I_{2m-1}(\omega) e^{j2\pi B (\frac{m-1}{m}) \alpha_{2m-1}} \} = F(\omega) . \quad (96)$$

A compact form of this equation is:

$$\sum_{l=-m}^{l=m-1} \sum_{k=0}^{k=2m-1} I_k(\omega) e^{j2\pi B \frac{l}{m} \alpha_k} F(\omega + 2\pi B \frac{l}{m}) = F(\omega) \quad (97)$$

Assuming independence, the requirement is, as in Equation (71) of Chapter VII, that all of the coefficients of $F(\omega + 2\pi B l/m)$ be zero except the coefficient of $F(\omega)$, which must be unity. Therefore

$$\begin{aligned} \sum_{k=0}^{k=2m-1} I_k(\omega) e^{j2\pi B \frac{l}{m} \alpha_k} &= 0, & l \neq 0, \\ &= 1, & l = 0. \end{aligned} \quad (98)$$

Solution of these equations, again assuming a solution exists, will give the $I_k(\omega)$ to which the samples must be applied (after having been passed through a low pass filter). The process of recovery is identical to that shown in Figure 14, if $f(nm/B + \alpha_k)$ is substituted for $f^k(nm/B)$. The filters, $I_k(\omega)$, are generally different. The filter requirements for negative frequency were discussed in Chapter VII.

Example:

Let samples of $f(t)$ be taken with period $1/B$ second. Assume that cluster sampling is used to reconstruct the function. How many sets of samples are needed, and when must they be taken? In this case $m = 1$, so two separate sample trains are needed. If the first is designated $f(n/B)$, the second can be designated $f(n/B + \alpha)$. α is arbitrary in the interval between 0 and $1/B$.

From Equation (93),

$$G_0(\omega) = F(\omega - 2\pi B) + F(\omega), \quad (99)$$

$$G_1(\omega) = e^{-j2\pi B\alpha} F(\omega - 2\pi B) + F(\omega).$$

Since it is required that

$$I_0(\omega) G_0(\omega) + I_1(\omega) G_1(\omega) = F(\omega), \quad (100)$$

then

$$I_0(\omega) \{F(\omega - 2\pi B) + F(\omega)\} + I_1(\omega) \{e^{-j2\pi B\alpha} F(\omega - 2\pi B) + F(\omega)\} = F(\omega). \quad (101)$$

Rearranging Equation (101),

$$F(\omega) \{I_0(\omega) + I_1(\omega)\} + F(\omega - 2\pi B) \{I_0(\omega) + e^{-j2\pi B\alpha} I_1(\omega)\} = F(\omega). \quad (102)$$

Therefore, from Equation (98)

$$I_0(\omega) + I_1(\omega) = 1, \quad (103)$$

$$I_0(\omega) + I_1(\omega) e^{-j2\pi B\alpha} = 0.$$

Whence

$$I_0(\omega) = \frac{e^{j(\pi B\alpha - \pi/2)}}{2 \sin \pi B\alpha} = I_0^*(-\omega), \quad (104)$$

$$I_1(\omega) = \frac{e^{j(\pi/2 - \pi B\alpha)}}{2 \sin \pi B\alpha} = I_1^*(-\omega).$$

As derived, this is valid in the positive frequency region $0 \leq \omega \leq 2\pi B$ only. However, if $I_0(\omega)$ and $I_1(\omega)$ are to represent physical filters, or filters that could be approximated physically, then $I_0(-\omega) = I_0^*(\omega)$ and $I_1(-\omega) = I_1^*(\omega)$. Since the sampled function is real, $G_k(-\omega) = G_k^*(\omega)$ and the output $F(-\omega)$ is therefore equal to $F^*(\omega)$. This was discussed in Chapter VII. Thus, if the $I_0(\omega)$ are determined so that $F(\omega)$ is recovered for positive frequency, $F(\omega)$ is recovered for negative frequency at the same time. (Interestingly enough, $I_0^*(\omega) = I_1(\omega)$ for this example.)

Examination of Equations (104) shows that $I_0(\omega)$ and $I_1(\omega)$ have the same magnitude and a constant (but different) phase shift. The replicas of $F(\omega)$ centered on $\omega = \omega_s = 2\pi B$ {denoted $F(\omega - 2\pi B)$ in Equations (99)} and the replicas of $F(\omega)$ centered in $\omega = 0$ in those equations are also equal in magnitude in both impulse trains. The replicas of $F(\omega)$ centered in $\omega = 0$ have the same relative phase in both trains. However, the replicas of $F(\omega)$ centered on $2\pi B$ rps have a phase difference of $2\pi B\alpha$ radians. If A is less than $B/2$, $I_0(\omega)$ and $I_1(\omega)$ advance and delay, respectively, the two signals so that the two replicas of $F(\omega)$ centered on $2\pi B$ rps are of opposite polarity at the output of the filters, and cancel when combined. The phase difference is thus used to separate the $F(\omega)$ from the other components.

If α is very small, the samples must be known with high accuracy.** When α is $B/2$, the sampling is of the type described in Chapter III and the transfer function of the filters $I_0(\omega)$ and $I_1(\omega)$ is unity.

**Yen (5) examines this type of sampling, the filters required, and the accuracy required for the samples. Black (17) mentions the method, but does not discuss it extensively.

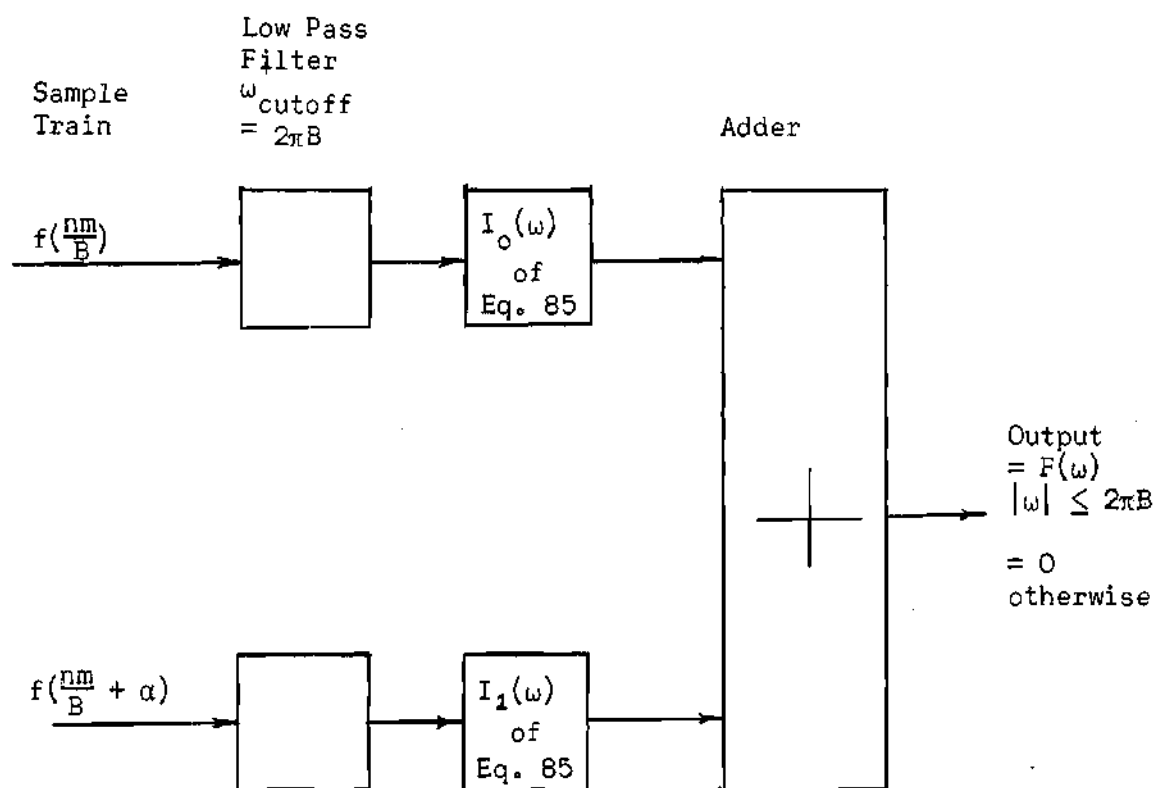


Figure 18. Recovery Method for Example of Chapter VIII

CHAPTER IX

CONCLUSIONS

A detailed study of constraints for sampling and recovery of a band-limited function has been presented for the following cases:

- (1) Samples of the function are taken periodically and at equally spaced intervals.
- (2) Samples of the function and its derivatives are taken periodically and at equally spaced intervals.
- (3) Samples of the function are taken periodically but at unequally spaced intervals.

Conditions pertinent to case (1) were considered in Chapters II through VI. A band-limited function was defined as one having a frequency spectrum $F(\omega)$ equal to zero for all $\omega > 2\pi B$ rps. The lowest value of B for which this is true was designated B_n . If such a function is sampled with non-zero time duration pulses, or if samples are taken instantaneously, it was shown that the function is completely recoverable if the samples are taken at a rate of $2B$ cps, where $B \geq B_n$. The frequency of $2B_n$ is commonly called the "Nyquist frequency."

Non-zero time duration sampling was developed by considering the multiplication of $f(t)$ by a pulse train $s(t)$, shown in Figure 2. This produced a new function of time designated $f_s(t)$, which was zero when $s(t)$ was zero and was $Af(t)$ when $s(t)$ was A . This new function was the set of non-zero time duration samples of $f(t)$.

Instantaneous sampling is an operation producing a train of impulses, each one having a value equal to the amplitude of $f(t)$ at

the instant of the impulse's occurrence. Physically, impulse samples are approximated by sampling a function of time with pulses whose duration approaches zero. This intuitive concept of impulses was used in Chapter IV, where it was assumed that the frequency spectrum of a train of instantaneous impulse samples was equal to the limit of a train of non-zero time duration samples, as pulse duration approached zero and amplitude became very large.

In both techniques of sampling, $f(t)$ was recovered by passing the samples through a low pass filter. The filter requirements for non-zero time duration samples were discussed in Chapter II, section 5. An ideal low pass filter of the type described in Chapter III, section 2, is required if the sampling is done at the Nyquist rate. The requirements become less strict as the sampling rate increases.

The filter requirements for impulse samples were discussed in Chapter III. Equation (24) was interpreted as giving the response of a low pass filter to an impulse train. On the basis of this interpretation, the function could be reconstructed if the instantaneous samples were applied to an ideal low pass filter having bandwidth half the sampling frequency, provided sampling was above the Nyquist rate.

The filter requirements for finite time duration samples, gotten from observation of the spectrum of the sampled function (Figure 4), were not uniquely related to the sampling rate. If any particular filter is adequate for recovery of $f(t)$ from a train of non-zero samples, that same filter would be adequate for any higher rate of sampling. The development for impulse sampling in Chapter III did not

rely on any concept of frequency spectrum, and the relationship between sampling rate and filter bandwidth is much more strict for such samples, according to a direct inspection of Equation (24). However, if the limiting operation of Chapter IV is valid, the sampling frequency can be arbitrarily increased above an adequate rate and the function still be recovered from those samples.

The constraints for sampling and recovery of a bandpass function were developed in Chapter V. Sufficient conditions were shown analytically and graphically. It was found that there are certain ranges of sampling frequencies below the Nyquist rate at which recovery is possible. These frequencies are functions of the bandwidth and lower frequency of the bandpass function.

A proof of the necessity of the Nyquist rate for instantaneous samples was presented. This proof does not include non-zero time duration samples.

Derivative and cluster sampling [cases (2) and (3)] were defined and treated in Chapters VII and VIII, respectively. Derivative sampling allows a function to be sampled below the Nyquist rate and still be uniquely determined if samples of the function's derivatives are also taken at a specified rate. Cluster sampling permits the samples of the function to be unequally spaced as long as an average of at least $2B_n$ per second are taken. In both cases $2B_n$ data of information per second are required.

The development of the constraints for determination and recovery was similar for derivative and cluster sampling. Sections of the spectrum

of the sampled function were shown to be the sum of several independent components determined by $F(\omega)$. The differences between the spectra of the samples of the original function and of the samples of the derivatives, or of the other samples in the cluster, were used to calculate the original $F(\omega)$ by means of simultaneous equations. Recovery was effected by passing the sample trains through appropriately designed networks.

If any other operations exist which make the replicas of $F(\omega)$ that are centered on various harmonics of ω_s in a particular train of samples different in some way from the replicas of $F(\omega)$ centered on the same harmonics of ω_s in another sample train, so that a set of equations having a unique solution may be developed, then a method involving these operations might be devised to recover $f(t)$ if it is sampled below the Nyquist rate. It is conjectured that at least $2B_n$ data per second still must be taken, however.

There are some implications of derivative and cluster sampling which would bear further study. If the values of a large number of derivatives of a function are available, then, according to Chapter VI, that function and its derivatives can be sampled well below the Nyquist rate and still be recovered. In the limit, as the number of derivatives approaches infinity, the value of the derivatives at one particular instant of time could conceivably determine the function for all time. But these could be taken from a single non-zero time duration pulse sample of $f(t)$.

A similar possibility exists with cluster sampling. If there are an average number of $2B_n$ separate samples per second, it was shown that the function could be determined. If non-zero time duration

samples are taken at a rate much slower than the Nyquist rate, but if each sample pulse is sampled so that an average of $2B$ instantaneous samples per second of the original function is obtained, the function is determined.

In both cases, finite time duration samples, which might be taken at as slow a rate as desired, intuitively appear to determine the function. Questions like these seem properly included in the problem of finding the information bearing elements or qualities of a signal.

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